University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #10

(Spring 2024)

Evaluate (by hand) the definite integral $\int_0^1 x^m(1-x)^n\,dx$.

Solution:

Solution. Choose u_1 through u_m, v_1 through v_n, and x independently at random with uniform distribution on the interval $[0,1]$.

Given any $x \in [0,1]$, the probability that $u_i < x$ is x and the probability that $v_j > x$ is $(1-x)$, so the joint probability that each u_i is less than x while each v_j is greater than x is $\int_0^1 x^m(1-x)^n\,dx$.

On the other hand, these $m+n+1$ variables can be ordered in $(m+n+1)!$ different permutations, all equally likely. Of these permutations, there are $m!\,n!$ that have all u_i's less than x and all v_j's greater than x, so $\int_0^1 x^m(1-x)^n\,dx = \boxed{\dfrac{m!\,n!}{(m+n+1)!}}$. □

Source: Richard Stanley, *Conversational Problem Solving*, American Mathematical Society (2010), 68–69.