UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Spring 2024)

Which prime numbers p have the property that $p^2 - 24p + 122$ is also prime?

Solution:
The only such primes are 3, 7, and 17.

Proof.
Let $f(p) = p^2 - 24p + 122$. Notice that $f(3) = 59$, which is prime, so $\boxed{p = 3}$ is a solution. Suppose from now on that $p \neq 3$. By Fermat's little theorem, we know (since 3 is prime and p is not a multiple of 3) that $p^2 - 1 = 3k$ for some integer k. [You can also check this yourself: the key idea is that $(3n \pm 1)^2 - 1 = 3(3n^2 \pm 2n)$.]
Now we know that

$$f(p) = (p^2 - 1) - 24p + 123 = 3(k - 8p + 41)$$

so $f(p)$ is a multiple of 3. Therefore, $f(p)$ is prime if and only if

$$\begin{aligned} f(p) &= 3 \\ p^2 - 24p + 122 &= 3 \\ p^2 - 24p + 119 &= 0 \\ (p-7)(p-17) &= 0 \\ &\boxed{p = 7 \text{ or } p = 17}. \end{aligned}$$

□

Source: Suggested by: Lokman Gökçe, "Quickies 1116," *Mathematics Magazine* **94**:5 (December 2021), 391, 398.