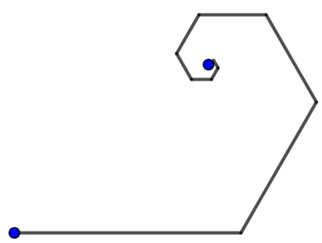
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Spring 2024)

When it gets switched on, a wind-up toy moves 3 cm in a straight line, then stops and makes a 60° left turn. Then it moves 2 cm in the new direction, after which it stops and makes another 60° left turn. It continues in the same way: each time it moves, it goes two-thirds as far as the time before, and in between movements it always turns 60° to the left. If this continues forever, the toy will approach a certain point. How far away is that point, in a straight line, from the toy's starting point?



Solution:
The distance is $\dfrac{9\sqrt{7}}{7}$ cm.

Proof. Suppose that the toy starts out at 0 in the complex plane, moving in the positive real direction. The displacement of the toy's first movement is 3; every future displacement is $\frac{2}{3}e^{\pi i/3} = \frac{1+i\sqrt{3}}{3}$ times the one before. The limit point at the end of the path is the sum of a geometric series:

$$\sum_{k=0}^{\infty} 3\left(\frac{1+i\sqrt{3}}{3}\right)^k = \frac{3}{1-\frac{1+i\sqrt{3}}{3}} = \frac{9}{2-i\sqrt{3}}.$$

The distance of this point from the origin is

$$\left|\frac{9}{2-i\sqrt{3}}\right| = \frac{|9|}{|2-i\sqrt{3}|} = \frac{9}{\sqrt{2^2+(-\sqrt{3})^2}} = \boxed{\frac{9}{\sqrt{7}}} = \boxed{\frac{9\sqrt{7}}{7}} \approx 3.4\,\text{cm}.$$

□

Remark. In standard form, the limiting position of the toy is

$$\frac{9}{2-i\sqrt{3}} \cdot \frac{2+i\sqrt{3}}{2+i\sqrt{3}} = \frac{18+9i\sqrt{3}}{7} \approx 2.57 + 2.23i.$$

Source: Problem 8, 2020 AIME I (American Invitational Mathematics Examination).