University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

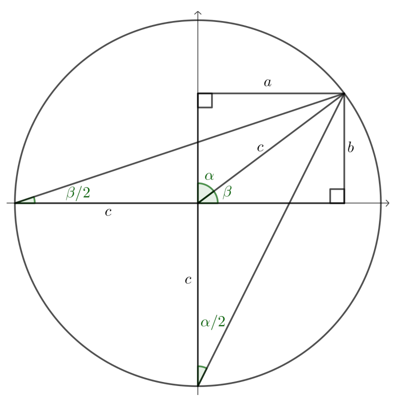
Problem of the Week #4

(Spring 2024)

Assume that $a^2 + b^2 = c^2$, for positive real numbers a, b, and c. Prove that

$$\arctan\left(\frac{a}{b+c}\right) + \arctan\left(\frac{b}{a+c}\right) = \frac{\pi}{4}.$$

Solution:



Proof. Draw a rectangle with sides of length a and b; by the Pythagorean theorem, the diagonal of this rectangle has length c. Let α and β be the angles opposite a and b respectively, so $\alpha + \beta = \frac{\pi}{2}$.

Draw a circle of radius c, with its center at one vertex of the rectangle, and inscribe angles in the circle that intercept the sides of the rectangle, as shown in the figure.

We know that the inscribed angles have measure $\alpha/2$ and $\beta/2$ respectively — this is the "inscribed angle theorem." These are acute angles, and we can read off their tangents from the figure: $\tan\frac{\alpha}{2} = \frac{a}{b+c}$ and $\tan\frac{\beta}{2} = \frac{b}{a+c}$. Therefore,

$$\arctan\left(\frac{a}{b+c}\right) + \arctan\left(\frac{b}{a+c}\right) = \frac{\alpha}{2} + \frac{\beta}{2} = \frac{1}{2}(\alpha + \beta) = \frac{\pi}{4}.$$

□

Source: Roger B. Nelson, "Pythagorean Triples and $\pi/4$," *Mathematics Magazine* **94**:3 (June 2021), 225.