University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #2

(Spring 2024)

In a certain network of sensors, each sensor station includes a screen meant to display the readings that are being observed at all stations. But the data on the display will go out of date if the sensors don't communicate.

Each station can connect to one other station at a time; it takes $1\,\mu$s for a pair of stations to connect, exchange all the information they currently have, then disconnect.

If the network has 4 stations $\{A, B, C, D\}$, then a complete network update takes $2\,\mu$s. For example, in the first microsecond, A and B can share their current observations while C and D do the same; then in the next microsecond, A can send C the previous microsecond's readings from A and B and receive the readings from C and D, while B and D make a similar exchange.

It might be a surprise that if the network has only 3 stations $\{A, B, C\}$, then a complete network update will take at least $3\,\mu$s. [Suppose A and B share information in microsecond 1; C has no partner with which to communicate at the same time, and can't send its reading to both A and B in microsecond 2.]

If the network has 10 stations, how quickly can a complete network update be performed?

Solution:

The fastest possible update takes $4\,\mu$s.

Inductively, any given station has fresh data from at most 2^n stations (including itself) after $n\,\mu$s, so no station can update all 10 stations' data within $3\,\mu$s, as $2^3 = 8 < 10$.

However, a complete update can be performed in $4\,\mu$s. For example, numbering the stations from 0 through 9, we can make the following connections:

1. $0 \leftrightarrow 1, 2 \leftrightarrow 3, 4 \leftrightarrow 5, 6 \leftrightarrow 7, 8 \leftrightarrow 9$
2. $0 \leftrightarrow 6, 1 \leftrightarrow 2, 3 \leftrightarrow 8, 4 \leftrightarrow 7, 5 \leftrightarrow 9$
3. $0 \leftrightarrow 8, 1 \leftrightarrow 4, 2 \leftrightarrow 9, 3 \leftrightarrow 7, 5 \leftrightarrow 6$
4. $0 \leftrightarrow 1, 2 \leftrightarrow 3, 4 \leftrightarrow 5, 6 \leftrightarrow 7, 8 \leftrightarrow 9$

Here's a table of the stations whose information each station has after each exchange:

0	1	2	3	4	5	6	7	8	9
01	01	23	23	45	45	67	67	89	89
0167	0123	0123	2389	4567	4589	0167	4567	2389	4589
01236789	01234567	01234589	23456789	01234567	01456789	01456789	23456789	01236789	01234589
all	all	all	all	all	all	all	all	all	all

Source: Dennis Shasha, *The Puzzling Adventures of Dr. Ecco*, W.H. Freeman and Company (1988), 62–64, 156.