University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Spring 2023)

Claudine has seven playing cards marked $1, 2, 3, 4, 5, 6, 7$. She shuffles the cards and deals three of them to Anne-Marie and three to Barb, keeping the remaining card for herself. All three people know the rules, and each can see the cards that they hold, but no others. Anne-Marie has never met or communicated with Barb before, so they have not agreed on a code or plan.

Explain how Anne-Marie can make one true public statement that will tell Barb exactly which cards Anne-Marie has, without letting Claudine learn the location of any card other than the one she holds herself.

Solution:

Anne-Marie can succeed by announcing the sum of the values on her cards modulo 7.

Proof. In this proof, all computations are done modulo 7.

Let A be the sum of Anne-Marie's cards' values, and B the sum of Barb's cards' values. If c is the value of Claudine's card, then

$$A + B + c \equiv 1 + \cdots + 7 = 28 \equiv 0.$$

Barb knows what cards she has, so she knows the value of B. When Anne-Marie announces A, Barb can compute $A + B$, which tells her the value of $c \equiv -(A + B)$. Now Barb knows which card Claudine has, and which cards she has herself; this tells Barb that Anne-Marie has the other three cards.

We still need to verify that Claudine can't learn the location of any card. Let z be a card with $z \neq c$. We will prove that, as far as Claudine knows, z could be in Anne-Marie's hand. Claudine knows A, B, and c, but nothing else.

Let h be the card with $h \equiv 4(A - z)$. Then $h + h = 2h \equiv 8(A - z) \equiv A - z$. The six other cards, besides h, can be arranged in three pairs $\{h-1, h+1\}$, $\{h-2, h+2\}$, and $\{h-3, h+3\}$, and the sum of each pair is $A - z$. It's possible that one pair includes c and another pair includes z, but even in that worst case, there is a third pair $\{x, y\}$ such that Anne-Marie's hand could be $\{x, y, z\}$.

The same argument, replacing A by B, shows that z could be in Barb's hand. Thus Claudine can't learn the location of z. □

Source: Daniel J. Velleman and Stan Wagon, *Bicycle or Unicycle?*, AMS/MAA Problem Books Volume 36. Providence: MAA Press (2020), 40, 239–240.