University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #8

(Spring 2023)

Simplify: $\sin^2\frac{\pi}{14}+\sin^2\frac{3\pi}{14}+\sin^2\frac{5\pi}{14}$.

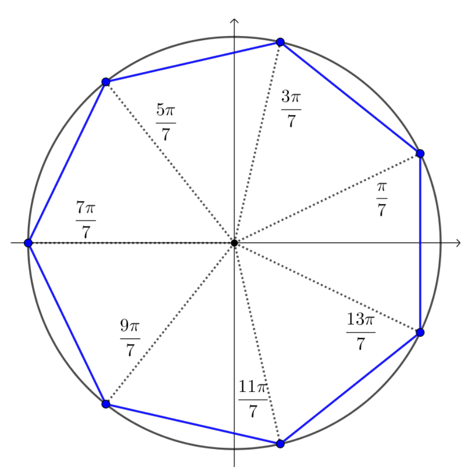
Solution:

By the power-reducing identity,

$$\begin{aligned}
&\sin^2\frac{\pi}{14}+\sin^2\frac{3\pi}{14}+\sin^2\frac{5\pi}{14}\\
&=\frac{1-\cos\frac{\pi}{7}}{2}+\frac{1-\cos\frac{3\pi}{7}}{2}+\frac{1-\cos\frac{5\pi}{7}}{2}\\
&=\frac{3}{2}-\frac{1}{2}\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\right)\\
&=\frac{3}{2}-\frac{1}{4}\left(\cos\frac{\pi}{7}+\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{5\pi}{7}\right)\\
&=\frac{3}{2}-\frac{1}{4}\left(\cos\frac{\pi}{7}+\cos\left(-\frac{\pi}{7}\right)+\cos\frac{3\pi}{7}+\cos\left(-\frac{3\pi}{7}\right)+\cos\frac{5\pi}{7}+\cos\left(-\frac{5\pi}{7}\right)\right)\\
&=\frac{3}{2}-\frac{1}{4}\left(\cos\frac{\pi}{7}+\cos\frac{13\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{11\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{9\pi}{7}\right)\\
&=\frac{3}{2}-\frac{1}{4}\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{9\pi}{7}+\cos\frac{11\pi}{7}+\cos\frac{13\pi}{7}\right)-\frac{1}{4}\cos\pi-\frac{1}{4}\\
&=\frac{3}{2}-\frac{1}{4}\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{7\pi}{7}+\cos\frac{9\pi}{7}+\cos\frac{11\pi}{7}+\cos\frac{13\pi}{7}\right)-\frac{1}{4}
\end{aligned}$$

Let H be the regular heptagon inscribed in the unit circle with a vertex at $(-1,0)$. The seven cosines in the sum above are the x-coordinates of the vertices of H. The average of those seven x-coordinates is the x-coordinate of the center of H, which is the origin. So the sum of the cosines is zero, and

$$\sin^2\frac{\pi}{14}+\sin^2\frac{3\pi}{14}+\sin^2\frac{5\pi}{14}=\frac{3}{2}-\frac{1}{4}=\boxed{\frac{5}{4}}.$$



Source: Art of Mathematics. "Solution – calculate the following." https://mathematicsart.com/solved-exercises/solution-sin-2-fracpi14sin-2-frac3-pi14sin-2-frac5-pi14/