UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Spring 2023)

I like to go for a walk in the evening, but I don't like to take the same route every night. The streets in my neighborhood are laid out in rectangular city blocks, and every time I get to a corner, I head in one of the four possible directions at random (including, sometimes, back the way I came — but I never turn around in the middle of a block).

My friend's house is on the corner, two blocks north and four blocks east of the corner where I live. If I start at my house, what is the probability that I'll reach my friend's corner by the time I've walked eight blocks?

Solution:

The probability is $\dfrac{157}{16{,}384} \approx 0.96\%$.

Proof. My friend's corner is six blocks from mine, so if I get there, it will be when I've walked exactly six or eight blocks.

A six-block path from my house to my friend's house must be made up of two north steps and four east steps, which can be ordered in $\binom{6}{2} = 15$ different ways. There are $4^6 = 4096$ possible six-block paths, so the probability that I'll reach my friend's corner when I've walked exactly six blocks is $P_1 = \frac{15}{4^6}$.

An eight-block path from my house to my friend's house must be made up of either three north steps, one south step, and four east steps, or else two north steps, one west step, and five east steps. There are $\binom{8}{3}\binom{5}{1} = 280$ paths of the first type and $\binom{8}{2}\binom{6}{1} = 168$ paths of the second type. There are 4^8 possible six-block paths, so the probability that I'll reach my friend's corner when I've walked exactly eight blocks is $P_2 = \frac{280+168}{4^8} = \frac{7}{4^5}$.

The probability that I'll reach my friend's corner is $P_1 + P_2 - P_3$, where P_3 is the "overlapping" probability that I get there after six blocks and then return after eight. Each of the 15 six-block paths to my friend's corner can be extended in four ways to an eight-block path that ends at my friend's corner, so $P_3 = \frac{60}{4^8} = \frac{15}{4^7}$, and the desired probability is

$$P_1 + P_2 - P_3 = \frac{15 \cdot 4 + 7 \cdot 4^2 - 15}{4^7} = \frac{157}{4^7}.$$

□

Source: Suggested by Problem #3, 1995 AIME. In: Scott A. Annin, *A Gentle Introduction to the American Invitational Mathematics Exam*, 46-47.