UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #6

(Spring 2023)

For each positive integer n, let D_n be the greatest odd divisor of n. (For example, $D_{168} = 21$.) Find $D_1 + D_2 + D_3 + \cdots + D_{2048}$.

Solution:
The sum equals 1,398,102.

Proof. Let $S_n = \sum_{k=1}^{2^n} D_k = D_1 + \cdots + D_{2^n}$; our goal is to find S_{11}, since $2^{11} = 2048$. Notice that $S_0 = D_1 = 1$.
If k is odd, then $D_k = k$; but for any even number $2k$, we have $D_{2k} = D_k$. Therefore,

$$\begin{aligned} S_{n+1} &= \sum_{k=1}^{2^{n+1}} D_k \\ &= \sum_{k=1}^{2^n} D_{2k-1} + \sum_{k=1}^{2^n} D_{2k} \\ &= \sum_{k=1}^{2^n} (2k-1) + \sum_{k=1}^{2^n} D_k \\ &= \sum_{k=1}^{2^n} (2k-1) + S_n. \end{aligned}$$

It is well known* that the odd integers from 1 through $2m-1$ add up to m^2, so

$$S_{n+1} = (2^n)^2 + S_n = 4^n + S_n.$$

Therefore,

$$\begin{aligned} S_{11} &= (S_{11} - S_{10}) + (S_{10} - S_9) + \cdots + (S_2 - S_1) + (S_1 - S_0) + S_0 \\ &= 4^{10} + 4^9 + \cdots + 4^1 + 4^0 + S_0 \\ &= \frac{4^{11} - 1}{4 - 1} + 1 \\ &= \frac{4^{11} + 2}{3} \\ &= \boxed{1{,}398{,}102}. \end{aligned}$$

□

Source: Titu Andreescu and Jonathan Kane. *Purple Comet! Math Meet: The First Ten Years.* XYZ Press (2013), 130-131.

*Among people who know about it.