UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Spring 2023)

Find every positive integer that equals four times the sum of its digits.

Solution:
The only such positive integers are $\{12, 24, 36, 48\}$.

Proof. Let x be a positive integer. If x has just one digit, then x equals the sum of its digits. If x has exactly two digits, we can write $x = 10d_1 + d_0$ with $0 \le d_i \le 9$. In this case, if x equals four times the sum of its digits, then

$$10d_1 + d_0 = 4(d_1 + d_0) \quad \Rightarrow \quad 6d_1 = 3d_0 \quad \Rightarrow \quad 2d_1 = d_0.$$

We can't have $x = 0$, so $0 < d_0 < 10$, and therefore $0 < d_1 < 5$. The four remaining possibilities are $x \in \{12, 24, 36, 48\}$, and each of these does equal four times the sum of its digits.
Now, for any positive integer x, let $f(x)$ denote x divided by the sum of the digits of x. If x is an M-digit number, then the decimal form of x is $x = \sum_{k=0}^{M-1} d_k \cdot 10^k$, and

$$f(x) = \frac{\sum_{k=0}^{M-1} d_k \cdot 10^k}{\sum_{k=0}^{M-1} d_k}.$$

We can read this as saying that x is the sum of a certain list of powers of ten, and $f(x)$ is the average (mean) value of that list. Since x has M digits, the list must contain at least one copy of 10^{M-1}.
Now assume $M \ge 3$. We know $f(10^{M-1}) = 10^{M-1}$. We can decrease the value of f by adding copies of $1 = 10^0$ to the list until we reach

$$f(10^{M-1} + 9) = \frac{10^{M-1} + 9}{1 + 9} = 10^{M-2} + \frac{9}{10} > 10.$$

At this point we can't increase d_0 any further. Increasing other digits of x adds numbers to the list that are at least 10, and that can't decrease the mean value $f(x)$ below 10. This proves that if x has three or more digits, then $f(x) > 10$, so $f(x) \ne 4$. $\square$