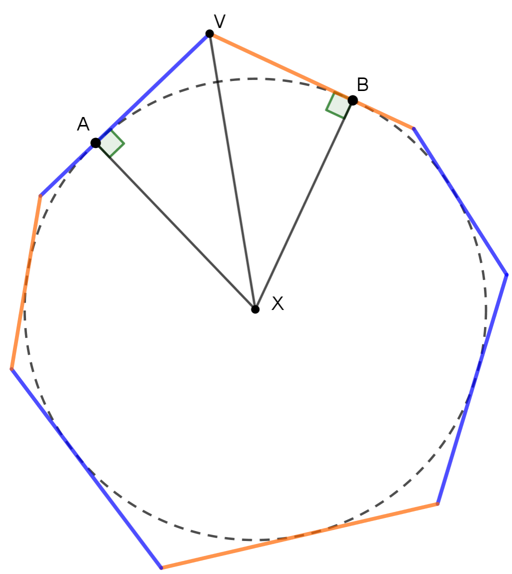
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #4

(Spring 2023)

Let P be a polygon whose edges have been painted either blue or orange, in such a way that the edges next to an orange edge are always blue. (However, blue edges may be next to each other.) Prove: if the sum of the lengths of the orange edges is greater than the sum of the lengths of the blue edges, then you can't inscribe a circle in P (that is, no circle is tangent to every edge of P).



Solution:

We prove the contrapositive claim: If you can inscribe a circle in P, then the sum of the lengths of the orange edges is less than or equal to the sum of the lengths of the blue edges. Let P be a polygon with orange and blue edges, with no two orange edges adjacent. Let C be a circle with center X, and suppose that C is inscribed in P.

Let V be a vertex of P. Two edges of P meet at vertex V; suppose that C is tangent to those edges at A and B. Radii of a circle are perpendicular to tangent lines, so $AV \perp AX$ and $BV \perp BX$. By applying the Pythagorean theorem to the right triangles $\triangle VAX$ and $\triangle VBX$, we see that VA and VB have equal length. However, VA and VB can't both be orange, and they might both be blue, so between A and B there is at least as much blue edge length as orange edge length.

Taking sums over all vertices of P, we find that the sum of the lengths of the orange edges is less than or equal to the sum of the lengths of the blue edges.

Source: Emina Soljanin, "Painting the Polyhedron," in: Peter Winkler, *Mathematical Puzzles: A Connoisseur's Collection*, A. K. Peters, Ltd. (2004), 45, 52.