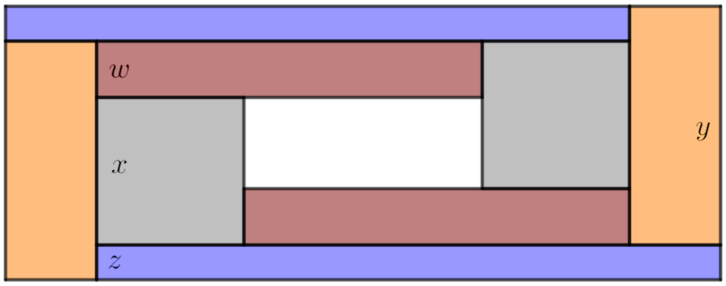
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

PROBLEM OF THE WEEK #3

(Spring 2023)

The rectangle shown below has been subdivided into nine rectangles of equal area. (Rectangles of the same color are congruent.) Find the value of x/y, where x is the height of the silver rectangles and y is the height of the orange rectangles.



Solution:

Let w and z be the heights of the maroon and blue rectangles respectively, and let A denote the common area of the nine rectangles. The largest and smallest rectangles have areas:

$$\begin{aligned}(y+z)\left(\tfrac{A}{y}+\tfrac{A}{z}\right) &= 9A\\ A\left(1+\tfrac{y}{z}+\tfrac{z}{y}+1\right) &= 9A\\ \tfrac{y}{z}+\tfrac{z}{y} &= 7\\ y^2+z^2 &= 7yz\\ (y-z)^2 &= 5yz\end{aligned}$$

$$\begin{aligned}(x-w)\left(\tfrac{A}{w}-\tfrac{A}{x}\right) &= A\\ A\left(\tfrac{x}{w}-1-1+\tfrac{w}{x}\right) &= A\\ \tfrac{x}{w}+\tfrac{w}{x} &= 3\\ x^2+w^2 &= 3xw\\ (x+w)^2 &= 5xw\end{aligned}$$

From the figure, $y-z = x+w$, so $5yz = 5xw$, or $\frac{y}{x} = \frac{w}{z}$. Now let $r = \frac{y}{x} = \frac{w}{z}$. Notice:

$$\begin{aligned}\tfrac{1}{x}+\tfrac{1}{y} &= \tfrac{1}{z}-\tfrac{1}{w}\\ \tfrac{yw}{x}+w &= \tfrac{yw}{z}-y\\ (r+1)w &= (r-1)y.\end{aligned}$$

So we have $y = rx$, $w = \frac{r-1}{r+1}\cdot y = \frac{r(r-1)}{r+1}\cdot x$, and $z = \frac{w}{r} = \frac{r-1}{r+1}\cdot x$. Using these substitutions,

$$\begin{aligned}x^2+w^2 &= 3xw\\ x^2+\tfrac{r^4-2r^3+r^2}{r^2+2r+1}\cdot x^2 &= 3\tfrac{r^2-r}{r+1}\cdot x^2\\ r^4-2r^3+2r^2+2r+1 &= 3r^3-3r\\ (r^2-4r-1)(r^2-r-1) \;=\; r^4-5r^3+2r^2+5r+1 &= 0\\ r = 2\pm\sqrt{5} \text{ or } r &= \tfrac{1\pm\sqrt{5}}{2}.\end{aligned}$$

But we need $r = \frac{y}{x} > 0$, which rules out two of these values. Also, $\frac{A}{w} > \frac{A}{x}$, since the white rectangle has positive width. This means $x > w$, and therefore $r(r-1) < r+1$, or $r^2-2r-1 < 0$. Hence $r \neq 2+\sqrt{5}$, so $r = \frac{1+\sqrt{5}}{2}$, and $\frac{x}{y} = \frac{1}{r} = \boxed{\frac{\sqrt{5}-1}{2}}$.

Source: Arsalan Wares, "Nine Rectangles," *Math Horizons* **30**:2 (November 2022), 30.