UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #2

(Spring 2023)

Suppose $\{a_1, a_2, a_3, \dots\}$ is a sequence of positive numbers with the property that for all $n \geq 2$, $(n+1)^{a_n} = n^{a_{n-1}}$.
For which values of n is $a_n < \dfrac{a_1}{2023}$?

Solution:

Let's rewrite the given identity:

$$\begin{aligned} (n+1)^{a_n} &= n^{a_{n-1}} \\ a_n &= \log_{n+1}\left(n^{a_{n-1}}\right) = a_{n-1}\log_{n+1} n. \end{aligned} \tag{1}$$

We claim that $a_n = a_1 \log_{n+1} 2$ for all $n \geq 2$.

Proof of claim.
The proof is by induction. Taking $n = 2$ in (1), we get $a_2 = a_1 \log_3 2$ as desired.
Assume for induction that $a_k = a_1 \log_{k+1} 2$. Then, taking $n = k+1$ in (1):

$$a_{k+1} = a_k \log_{k+2}(k+1) = (a_1 \log_{k+1} 2)\log_{k+2}(k+1) = a_1 \frac{\ln 2}{\cancel{\ln(k+1)}} \cdot \frac{\cancel{\ln(k+1)}}{\ln(k+2)} = a_1 \log_{k+2} 2,$$

completing the induction. □

Now that we have a formula for a_n:

$$\begin{aligned} & a_n &<& \tfrac{a_1}{2023} \\ \iff & \cancel{a_1}\log_{n+1} 2 &<& \tfrac{\cancel{a_1}^{\,1}}{2023} \qquad \text{(since } a_1 > 0\text{)} \\ \iff & 2 &<& (n+1)^{1/2023} \\ \iff & 2^{2023} &<& n+1 \\ \iff & 2^{2023} &\leq& n. \end{aligned}$$

Source: Dan Swenson, Black Hills State University.