University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #1

(Spring 2023)

Let S be the real number written in decimal notation as $0.02040608101214\ldots$. Specifically, $S = \frac{2}{10^2} + \frac{4}{10^4} + \frac{6}{10^6} + \ldots$, with terms overlapping in the decimal form once the numerators have three or more digits.

Write S as a fraction in lowest terms.

Solution:

$S = \frac{200}{9801}$.

Proof.

$$\begin{array}{rcl} 100S &=& 2.0406081012\ldots \\ S &=& 0.0204060810\ldots \\ \hline 99S &=& 2.0202020202\ldots \end{array} \qquad \begin{array}{rcl} 9900S &=& 202.0202020202\ldots \\ 99S &=& 2.0202020202\ldots \\ \hline 9801S &=& 200 \end{array}$$

So $S = \frac{200}{9801}$. □

Alternate proof. We have $S = f(1/10)$, where

$$\begin{aligned} f(x) &= 2x^2 + 4x^4 + 6x^6 + \ldots \\ \frac{f(x)}{x} &= 2x + 4x^3 + 6x^5 + \ldots \\ \frac{f(x)}{x} &= \frac{d}{dx}\left[x^2 + x^4 + x^6 + \ldots\right] \\ \frac{f(x)}{x} &= \frac{d}{dx}\left[\frac{x^2}{1-x^2}\right] \qquad \text{(geometric series sum for } |x| < 1) \\ \frac{f(x)}{x} &= \frac{(1-x^2)(2x) - (x^2)(-2x)}{(1-x^2)^2} \\ \frac{f(x)}{x} &= \frac{2x}{(1-x^2)^2} \\ f(x) &= \frac{2x^2}{(1-x^2)^2} \end{aligned}$$

Since $\left|\frac{1}{10}\right| < 1$, we have $S = f\left(\frac{1}{10}\right) = \frac{2/100}{(99/100)^2} = \frac{2}{100} \cdot \frac{100^2}{99^2} = \frac{200}{9801}$. □