University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #10

(Spring 2022)

Let $x = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$, where n is a positive integer. Since x is a rational number, it has a repeating or terminating decimal form. Show that x has a repeating (non-terminating) decimal, but the repeating part doesn't start immediately after the decimal point.
For example, when $n = 4$, we get $x = \frac{37}{60} = 0.616666\ldots$, in which the "61" that follows the decimal point doesn't repeat.

Solution:

Proof. When we find a common denominator, we get $x = \frac{3n^2 + 6n + 2}{n(n+1)(n+2)}$. Suppose $x = \frac{a}{b}$ in lowest terms.
The factors n, $n+1$, and $n+2$ in the denominator of x are three consecutive integers, so one of them is a multiple of 3. Since $3n^2 + 6n + 2$ is one less than a multiple of 3, b (the lowest-terms denominator of x) is a multiple of 3. Thus (by unique prime factorization) x can't be written with a power of 10 as its denominator, so the decimal form of x doesn't terminate.
If n is odd, then the numerator of x is also odd, but the denominator (a product of consecutive integers) is even, so b is even. Otherwise, $n = 2k$ for some $k \in \mathbb{Z}$, and $x = \frac{6k^2 + 6k + 1}{k(2k+1)(2k+2)}$. In this form, we again have an odd numerator and an even denominator, so b is even in this case too. The fact that b is even implies that the periodic part of the decimal form of x does not start immediately after the decimal point.
To see this, let y be any rational number with a repeating decimal that begins immediately after the decimal point. Let k be the length of the repeating block of digits in the decimal form of y. Then $c = 10^k y - y$ is a whole number, because the digits after the decimal point cancel in the subtraction. Thus we can write $y = \frac{c}{10^k - 1}$ as a fraction with an odd denominator, so the denominator of y in lowest terms is also odd. (More specifically, it's relatively prime to 10.) □

Source: D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom. *The USSR Olympiad Problem Book.* Mineola: Dover Publications (1993), 17, 142–143.