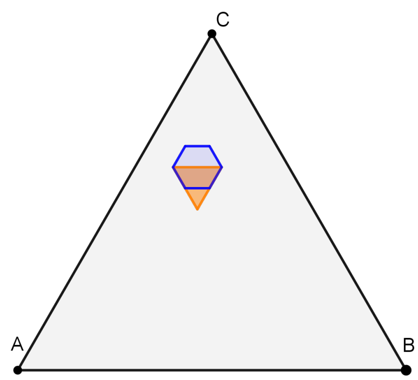
University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Spring 2022)

Let $\triangle ABC$ be an equilateral triangle with sides of length L. A “downward triangle” is an equilateral triangle with sides of length 1, having edges parallel to the edges of $\triangle ABC$, but with the opposite orientation, as shown in the figure.
Prove: If n downward triangles can fit inside $\triangle ABC$, without overlapping except along edges, then $n \leq \frac{2}{3}L^2$.



Solution:

Proof. Thinking of edge $\overline{AB}$ as the base of $\triangle ABC$, we refer to the edge parallel to $\overline{AB}$ in each downward triangle as a “top edge.”
On each downward triangle T, construct a “top hexagon” — a regular hexagon with opposite vertices at the endpoints of the top edge of T. Two downward triangles are disjoint if and only if their top hexagons are disjoint. If T lies inside $\triangle ABC$, then so does its top hexagon. Each downward triangle can be decomposed into 4 equilateral triangles with sides of length $\frac{1}{2}$, and each top hexagon can be decomposed into 6 such triangles. So if each downward triangle has area A, then each top hexagon has area $\frac{3}{2}A$, while by similarity, $\triangle ABC$ has area AL^2.
Therefore, if n disjoint downward triangles fit inside $\triangle ABC$ without overlapping, then $n \cdot \frac{3}{2}A \leq AL^2$, so $n \leq \frac{2}{3}L^2$. $\square$

Source: Béla Bajnow and Evan Chen. “Report on the 12th Annual USA Junior Mathematical Olympiad.” *College Mathematics Journal* **53**:1 (January 2022), 13–20.