UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Spring 2022)

Oneil Cruz, a shortstop for the Pittsburgh Pirates, had an OPS of 1.000 in Major League Baseball's 2021 season, according to MLB.com. The OPS statistic is a sum of two fractions: the on-base percentage and the slugging percentage. For Cruz, it looks like this sum is exactly equal to 1 — but it might not be, since the website rounds OPS to three decimal places.
Find the smallest integer N for which there are positive integers a, b, c, and d such that $b \leq N$, $d \leq N$, and $\left|\frac{a}{b}+\frac{c}{d}-1\right|$ is in the open interval $(0, 0.0005)$.

Solution:

The smallest possible value is $N = 46$.

Proof. Taking $(a,b,c,d) = (1,45,45,46)$, we have:

$$\left|\frac{a}{b}+\frac{c}{d}-1\right| = \frac{1}{45}+\frac{45}{46}-1 = \frac{46+45^2-45\cdot 46}{45\cdot 46} = \frac{1}{2070} \in \left(0, \frac{1}{2000}\right).$$

On the other hand, suppose that $0 < \left|\frac{a}{b}+\frac{c}{d}-1\right| < \frac{1}{M}$ with $M \geq 2$ (in fact, we will take $M = 2000$), and suppose without loss of generality that $b \leq d$. Finding a common denominator, we have $0 < \frac{|ad+bc-bd|}{bd} < \frac{1}{M}$, and since the numerator must be a positive integer, we have $\frac{1}{bd} < \frac{1}{M}$, which means $bd > M$. We consider two cases. First, if $b < d$, then $d(d-1) \geq db > M$. On the other hand, if $b = d$, then:

$$\begin{aligned} \frac{|ad+bc-bd|}{bd} &< \frac{1}{M} \\ \frac{|a+c-d|}{d} &< \frac{1}{M} \\ \frac{1}{d} &< \frac{1}{M} \\ d &> M. \end{aligned}$$

Now $d-1 > M-1 \geq 2-1 = 1$, so $d(d-1) > d > M$.
In either case $d(d-1) > M$. Taking $M = 2000$, we have $d^2 - d - 2000 > 0$, which means that either $d < \frac{1-\sqrt{8001}}{2} \approx -44.2$ or $d > \frac{1+\sqrt{8001}}{2} \approx 45.2$. Since d is a positive integer, $d \geq 46$. □

Source: Prof. Dan Swenson (Black Hills State University).