University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #2

(Spring 2022)

Let $f(x) = x^9 - 6x^7 + 9x^5 - 4x^3$. Show that if x is an integer, then $f(x)$ is divisible by 8640.

Solution:

$$\begin{aligned} f(x) &= x^3(u^3 - 6u^2 + 9u - 4) \qquad (u = x^2) \\ &= x^3(u-1)(u^2 - 5u + 4) \\ &= x^3(u-1)^2(u-4) \\ &= x^3(x-1)^2(x+1)^2(x-2)(x+2). \end{aligned}$$

$$\begin{array}{r|rrrr} & 1 & -6 & 9 & -4 \\ 1 & & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

Therefore:

- Among any four consecutive integers, there is a multiple of 4 and another multiple of 2. Since $f(x)$ is a multiple of $[(x-2)(x-1)(x)(x+1)][(x-1)(x)(x+1)(x+2)]$, $f(x)$ is divisible by $8^2 = 2^6$.
- Among any three consecutive integers, there is a multiple of 3. Since

  $$f(x) = [(x-2)(x-1)(x)] \cdot [(x-1)(x)(x+1)] \cdot [(x)(x+1)(x+2)],$$

  $f(x)$ is divisible by 3^3.
- Among any five consecutive integers, there is a multiple of 5. Since $f(x)$ is a multiple of $(x-2)(x-1)(x)(x+1)(x+2)$, $f(x)$ is divisible by 5.

Since 2^6, 3^3, and 5 are relatively prime, $f(x)$ is divisible by $2^6 \cdot 3^3 \cdot 5 = 8640$.

Source: Charles W. Trigg. "127. Quantity Divisible by 8640." *Mathematical Quickies: 270 Stimulating Problems with solutions.* New York: Dover Publications, Inc. (2013), 36, 135.