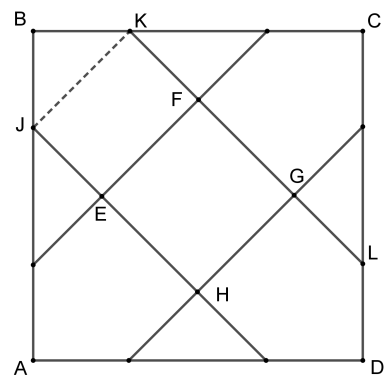
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #1

(Spring 2022)

The eight segments that form the figure below are all equally long, and the angles formed by their intersections are all exact multiples of 45°. If a point in the square $ABCD$ is chosen at random (uniformly), find the probability that the chosen point lies in the square $EFGH$.



Solution:

Label J, K, and L as in the figure. Let x and y denote the side lengths of the inner square and the outer square respectively; the desired probability is the area ratio $P = \frac{x^2}{y^2}$. Since FG and EH are parallel, $|JK| = |EF| = x$. Then, because $\triangle BKJ$ is an isosceles right triangle with hypotenuse x, $|BK| = \frac{x}{\sqrt{2}}$. Likewise, $\triangle CLK$ is an isosceles right triangle with hypotenuse y, so $|KC| = \frac{y}{\sqrt{2}}$. Now we have

$$\begin{aligned}
|BC| &= |BK| + |KC| \\
y &= \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\
y\sqrt{2} &= x + y \\
y(\sqrt{2} - 1) &= x \\
y^2(2 - 2\sqrt{2} + 1) &= x^2 \\
P = \frac{x^2}{y^2} &= \boxed{3 - 2\sqrt{2}} \approx 0.1716.
\end{aligned}$$

Source: Titu Andreescu and Jonathan Kane. *Purple Comet! Math Meet: The first ten years.* XYZ Press (2013), 204-5.