University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Spring 2021)

Let $f(x)$ and $g(x)$ be polynomials with rational coefficients, with $g(x) \neq 0$. Suppose that there are infinitely many integers a for which $\dfrac{f(a)}{g(a)}$ is an integer. Show that $f(x)$ is a multiple of $g(x)$; in other words, $f(x) = g(x)q(x)$, where $q(x)$ is a polynomial with rational coefficients.

Solution:

Proof. By long division, we can write $f(x) = q(x)g(x) + r(x)$, where q and r are polynomials with rational coefficients and $\deg r < \deg g$. Let d be a common denominator for the coefficients of $q(x)$, so that $dq(x)$ has integer coefficients.

Now suppose a is an integer and $\dfrac{f(a)}{g(a)}$ is an integer. Then $d \cdot \dfrac{f(a)}{g(a)}$ is an integer, as is $dq(a)$. Thus their difference,

$$d \cdot \frac{f(a)}{g(a)} - dq(a) = d \cdot \frac{q(a)g(a) + r(a)}{g(a)} - d \cdot \frac{q(a)g(a)}{g(a)} = d \cdot \frac{r(a)}{g(a)}$$

is an integer as well.

But because $\deg r < \deg g$, we know $\lim_{x\to\infty} \dfrac{r(x)}{g(x)} = 0$, so eventually $\left| d \cdot \dfrac{r(a)}{g(a)} \right| < 1$. If $\dfrac{f(a)}{g(a)}$ is an integer for infinitely many integers a, then $d \cdot \dfrac{r(a)}{g(a)} = 0$ for all but finitely many of them.

Then r is a polynomial with infinitely many zeros, so $r(x) = 0$, and therefore $\dfrac{f(x)}{g(x)} = q(x)$, a polynomial with rational coefficients. $\square$

Source: Titu Andreescu and Zuming Feng. *USA and International Mathematical Olympiads 2002.* Washington: The Mathematical Association of America (2003), 44–45.