University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #9

(Spring 2021)

Suppose $x_1 = 0$, $x_2 = 1$, and when $n \geq 1$, $x_{n+2} = x_{n+1} + \dfrac{x_n}{n}$. Find $\displaystyle\lim_{n\to\infty} \frac{n}{x_n}$.

Solution:

The limit is e.

Proof. For $n \geq 2$, let $d_n = \dfrac{x_n}{n} - \dfrac{x_{n-1}}{n-1}$. We claim that $d_n = \dfrac{(-1)^n}{n!}$. The proof is by induction on n: $d_2 = \dfrac{x_2}{2} - x_1 = \dfrac{1}{2}$, and for $n \geq 3$,

$$\begin{aligned} d_n &= \frac{x_n}{n} - \frac{x_{n-1}}{n-1} \\ &= \frac{1}{n}\left(x_{n-1} + \frac{x_{n-2}}{n-2}\right) - \frac{x_{n-1}}{n-1} \\ &= \frac{1}{n}\left[\frac{(n-1)x_{n-1}}{n-1} + \frac{x_{n-2}}{n-2} - \frac{nx_{n-1}}{n-1}\right] \\ &= -\frac{1}{n}\left[\frac{x_{n-1}}{n-1} - \frac{x_{n-2}}{n-2}\right] \\ &= -\frac{1}{n}d_{n-1}. \end{aligned}$$

The d_n are the terms of a telescoping sum: $\displaystyle\sum_{k=2}^{n} d_k = \frac{x_n}{n} - \frac{x_1}{1} = \frac{x_n}{n}$. On the other hand,

$$\sum_{k=2}^{n} d_k = \sum_{k=2}^{n} \frac{(-1)^k}{k!} = 1 - 1 + \sum_{k=2}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

which gives us

$$\lim_{n\to\infty} \frac{n}{x_n} = \frac{1}{\lim_{n\to\infty} \frac{x_n}{n}} = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}} = \frac{1}{e^{-1}} = e.$$

$\square$

Source: Daniel J. Velleman and Stan Wagon. *Bicycle or Unicycle?* Providence: MAA Press (2020), 65, 189.