University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #8

(Spring 2021)

Find every set $\{(a,p),(b,q)\}$ of distinct ordered pairs of positive integers such that, for all n,

$$\left[\sum_{i=1}^{n} i^a\right]^p = \left[\sum_{j=1}^{n} j^b\right]^q.$$

Solution:

The only such sets are $\{(1,2q),(3,q)\}$ (for any integer q).

Proof. It is well known that for any n and q,

$$(1+2+\cdots+n)^{2q} = \left(\frac{n(n+1)}{2}\right)^{2q} = \left(\frac{n^2(n+1)^2}{4}\right)^q = (1^3+2^3+\cdots+n^3)^q.$$

Conversely, suppose $\left[\sum_{i=1}^n i^a\right]^p = \left[\sum_{j=1}^n j^b\right]^q$ for every n. We can assume that p and q are relatively prime; if not, take the d^{th} root of both sides of the equation, where $d = \gcd\{p,q\}$. Suppose without loss of generality that $a < b$. Taking $n = 2$,

$$\begin{aligned}
(1^a+2^a)^p &= (1^b+2^b)^q \\
\sum_{s=0}^{p} \binom{p}{s} 2^{as} &= \sum_{t=0}^{q} \binom{q}{t} 2^{bt} \\
1 + p2^a &\equiv 1 \pmod{2^{a+1}} \\
p2^a &\equiv 0 \pmod{2^{a+1}}
\end{aligned}$$

Thus p is even (say, $p = 2\ell$), and because p and q are relatively prime, q is odd (say, $q = 2m+1$). Looking back to our equation $(1^a+2^a)^p = (1^b+2^b)^q$, each side must be a perfect square (because p is even), so $1+2^b$ is a perfect square (because q is odd). Fix $r \in \mathbb{Z}$ with $1+2^b = r^2$. Then $2^b = (r-1)(r+1)$, which means that $r-1$ and $r+1$ are powers of two that differ by 2. Hence $r = 3$, and so $b = 3$.

$$\begin{aligned}
(1^a+2^a)^p &= (1^3+2^3)^q \\
(1+2^a)^{2\ell} &= 9^{2m+1} \\
(1+2^a)^{\ell} &= 9^m \cdot 3 \\
(1+2^a)^{\ell} &\equiv 1^m \cdot 3 \;=\; 3 \pmod{4}.
\end{aligned}$$

But $(1+2^0)^\ell = 2^\ell \equiv \begin{cases} 2 & \ell = 0, \\ 0 & \ell \geq 1, \end{cases}$ and $(1+2^a)^\ell \equiv 1^\ell = 1$ when $a \geq 2$, so $a = 1$. Therefore $p = 2q$. $\square$

Source: Seljon Akhmedli. "The Carousel — Oldies But Goodies: P.S.P.I (C31)." *Math Horizons* **28**:2 (November 2020), 31, 33.