University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Spring 2021)

The toy shown here is the Fisher-Price Giant Rock-a-Stack. It has six colored rings with holes of different sizes that fit over a tapered post. When you drop a ring onto the post, it slides down until it either reaches its "proper" place or lands on another ring that's already on the post. Rings won't stay on the post if they're stacked above the smallest ring's proper place.

How many different stacks can be formed on the post using at least one ring?

Image: https://smile.amazon.com/Fisher-Price-GJW15-Giant-Rock-A-Stack/dp/B07PFYWVDQ/

Solution:

There are exactly 209 possible stacks.

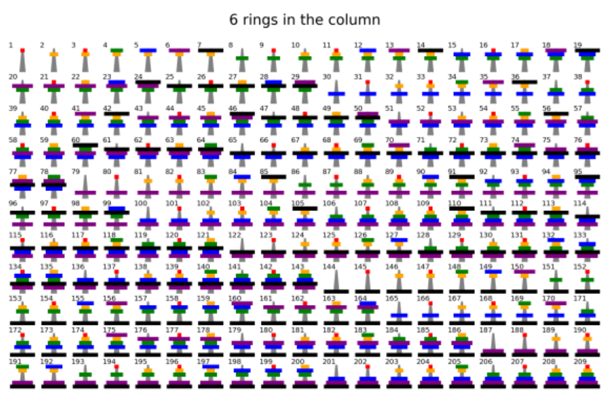

Image: Allen Gu, [https://fivethirtyeight.com/features/can-the-hare-beat-the-tortoise/]

Proof. Let's count the possible stacks that use exactly k rings, where $1 \le k \le 6$. The first ring to go onto the post can't land in one of the top $k-1$ positions, so it can't be one of the smallest $k-1$ rings. Any of the other $6-(k-1) = 7-k$ rings that can be used first. The second ring placed can't be one of the $k-2$ smallest rings. Nor can it be the ring which was placed first (which wasn't one of the $k-2$ smallest ones). Any of the other $6-(k-2)-1 = 7-k$ rings can be placed next. More generally, for $1 \le n \le k$, the n^{th} ring placed can't be one of the $k-n$ smallest rings, and can't be one of the $n-1$ rings that have already been placed, so there are $6-(k-n)-(n-1) = 7-k$ choices for the n^{th} ring.

Thus there are $(7-k)^k$ possible stacks that use exactly k rings, and altogether there are

$$\sum_{k=1}^{6}(7-k)^k = 6^1 + 5^2 + 4^3 + 3^4 + 2^5 + 1^6 = \boxed{209}$$

possible stacks. □

Source: Zach Wissner-Gross. "Can You Make 24?" *The Riddler*, 10 July 2020. Available at [https://fivethirtyeight.com/features/can-you-make-24/].