University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #5

(Spring 2021)

I've designed two robots to play a game on an $m \times n$ grid of squares. The "guard" robot starts the game by marking each square on the grid with an arrow pointing at one of the eight neighboring squares, in such a way that the arrows on a pair of neighboring squares never differ in direction by more than 45°. Then the "runner" robot starts from a randomly selected square on the grid and follows the arrows from square to square.
Show that the runner will eventually reach a square on the edge of the grid.

Solution:

Proof. Suppose for contradiction that the runner never reaches the edge. There are a finite number of squares on the grid, so the runner must eventually revisit a square. From then on, the runner repeatedly follows a closed loop L.
The runner never crosses over its own path. On a grid of squares, the runner could only cross its path by passing over a certain vertex while moving from a square x to its diagonal neighbor, and then later passing over the same vertex while moving from y to its diagonal neighbor. But in this case, x and y would share an edge, and their arrows would differ in direction by 90°, which is not allowed.
So the loop L has an inside and an outside; let M ("middle") denote the set of squares strictly inside L. Now the guard can become more strict by rotating every arrow on the board by 45°, clockwise if the runner follows L clockwise and vice versa. On the new board, every square on L is marked with an arrow that points to a square in M, and it is still true that two neighboring arrows never differ in direction by more than 45°. If the runner starts from a square on L and follows the new arrows, it will move immediately into M, and will never move outside of L. Thus the runner will eventually follow a loop L' whose inside M' is strictly smaller than M.
The guard can repeat this relabelling process indefinitely, and if it does so, the sizes of the "inside" sets will form an infinite decreasing sequence of non-negative integers, which is impossible. □

Source: Kevin Purbhoo. Published in: Winkler, Peter. "Lemming on a Chessboard." *Mathematical Mind-Benders.* Wellesley: A K Peters, Ltd. (2007), 67, 73.