University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #4

(Spring 2021)

Dr. Black owns 8 different pairs of socks. When he does laundry, paired socks are always washed together, and the number of pairs of socks in a load follows a uniform distribution between 0 and 8 pairs.
The other day Dr. Black was pulling socks out of the dryer and noticed that he had pulled out 6 single socks without completing a pair! What is the probability there were exactly 6 pairs of socks in this load?

Solution:

The probability is $\frac{13}{83}$.

Proof. Assume that socks are removed from the dryer one at a time. Let A_n be the event that exactly n pairs of socks were initially in the dryer, and let B denote the event that the first six socks removed from the dryer are from different pairs. We want to compute $P(A_6|B)$, and we are given that $P(A_n) = \begin{cases} \frac{1}{9}, & 0 \le n \le 8; \\ 0, & \text{else.} \end{cases}$
By the pigeonhole principle, $P(B|A_n) = 0$ when $n < 6$. Assuming that $6 \le n \le 8$, we find

$$P(B|A_n) = \frac{(2n)(2n-2)\dots(2n-10)}{(2n)(2n-1)\dots(2n-5)} = \frac{2^6 \cdot n!/(n-6)!}{(2n)!/(2n-6)!} = \frac{2^6(n)!(2n-6)!}{(2n)!(n-6)!}.$$

Thus

$$P(B) = \sum_n P(B|A_n)P(A_n) = \frac{1}{9}\sum_{n=6}^{8} P(B|A_n) = \frac{1}{9}\left[\frac{16}{231} + \frac{64}{429} + \frac{32}{143}\right] = \frac{1328}{27027}.$$

By Bayes's theorem,

$$P(A_6|B) = \frac{P(B|A_6)P(A_6)}{P(B)} = \frac{\left(\frac{16}{231}\right)\left(\frac{1}{9}\right)}{\left(\frac{1328}{27027}\right)} = \boxed{\frac{13}{83}} \approx 15.66\%.$$

□

Source: Prof. Michael Black.