University of Wisconsin
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Problem of the Week #2

(Spring 2021)

Suppose that a, b, and c are positive integers with

$$c = (a+bi)^3 - 107i$$

(where as usual $i^2 = -1$). Find c.

Solution:

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$$c = (a+bi)^3 - 107ic = a^3 + 3a^2bi - 3ab^2 - b^3i - 107i$$
$$c + 0i = (a^3 - 3ab^2) + (3a^2b - b^3 - 107)$$
$$\begin{cases} c & = a^3 - 3ab^2 \\ 0 & = 3a^2b - b^3 - 107 \end{cases}$$
$$107 = b(3a^2 - b^2)$$

Hence $b \mid 107$, which is prime, so $b = 1$ or $b = 107$.
We claim that $b = 1$. Suppose, on the contrary, that $b = 107$. Then $3a^2 - b^2 = 1$, so

$$3a^2 = 1 + b^2 = 1 + 107^2 = 11450,$$

which is impossible since $3 \nmid 11450$.
Since $b = 1$, we have:

$$\begin{aligned} 107 &= 3a^2 - 1 \\ 108 &= 3a^2 \\ 36 &= a^2 \\ 6 &= a \\ c &= 6^3 - 3(6)(1)^2 \\ &\boxed{c = 198}. \end{aligned}$$

□

Source: 1985 AIME, Problem #3. In Annin, Scott. *A Gentle Introduction to the American Invitational Mathematics Exam* (2015), pp. 151, 315.