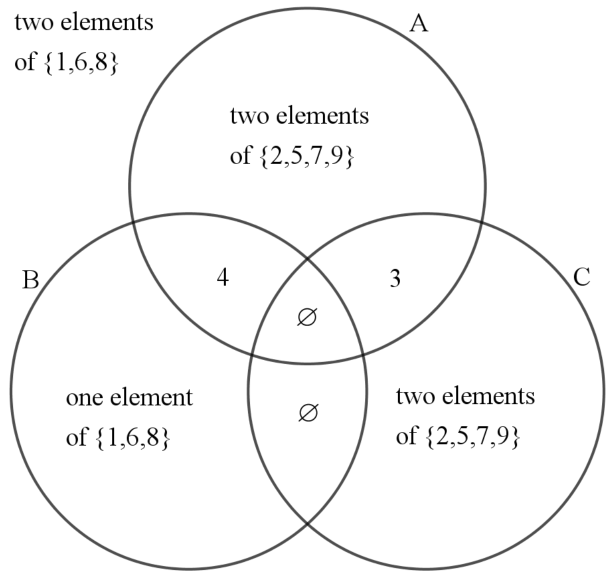
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #1

(Spring 2021)

Let $U = \{1,2,3,4,5,6,7,8,9\}$, and let S' denote $U - S$, the complement of S in U.
In how many ways can we find subsets A, B, and C of U with the following six properties?

$$\begin{aligned} A \cap B &= \{4\} \\ A \cap C &= \{3\} \\ B \cap C &= \varnothing \\ A \cup C &= \{2,3,4,5,7,9\} \\ |A \cap B'| &= 3 \\ |(A \cup B \cup C)'| &= 2 \end{aligned}$$



Solution:

There are 18 solutions.

Proof. First,

$$\{1,6,8\} = (A \cup C)' = A' \cap C' = (A' \cap B \cap C') \sqcup (A' \cap B' \cap C').$$

Since $|A' \cap B' \cap C'| = |(A \cup B \cup C)'| = 2$, $A' \cap B' \cap C'$ is a 2-element subset of $\{1,6,8\}$, and the other element of $\{1,6,8\}$ is the only element of $A' \cap B \cap C'$.
We know $A \cap B \cap C' \subseteq A \cap B = \{4\}$. On the other hand, since $B \cap C = \varnothing$, we know $4 \notin C$, so $4 \in A \cap B \cap C'$. Thus $A \cap B \cap C' = \{4\}$.
Likewise, $A \cap B' \cap C \subseteq A \cap C = \{3\}$. On the other hand, since $B \cap C = \varnothing$, we know $3 \notin B$, so $3 \in A \cap B' \cap C$. Thus $A \cap B' \cap C = \{3\}$. Since $|A \cap B'| = 3$, we conclude that $|A \cap B' \cap C'| = 2$.
It follows that $(A \cap B' \cap C') \sqcup (A' \cap B' \cap C) = \{2,5,7,9\}$. Thus $A \cap B' \cap C'$ is a 2-element subset of $\{2,5,7,9\}$, and the other two elements of $\{2,5,7,9\}$ are the only elements of $A' \cap B' \cap C$.

Conversely: Let S be any one-element subset of $\{1,6,8\}$ and let T be any two-element subset of $\{2,5,7,9\}$, with $U = \{2,5,7,8\} - T$. Then

$$(A,B,C) = (T \cup \{3,4\}, S \cup \{4\}, U \cup \{3\})$$

satisfies all of the given properties. There are $\binom{3}{1} = 3$ choices for S and $\binom{4}{2} = 6$ choices for T, so there are $3 \cdot 6 = 18$ ways to choose (A,B,C) subject to the given requirements. $\square$

Source: Holly Attenborough.