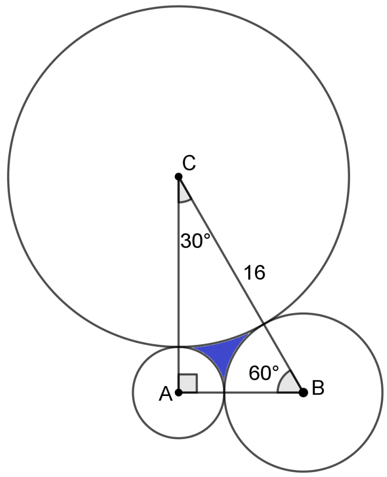
University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #6

(Spring 2020)

Three circles are externally tangent to each other, and their centers are the vertices of a $30 - 60 - 90$ triangle T whose hypotenuse has length 16. Find the area of the region that is inside T but outside all three circles.



Solution:
The area is $\frac{1}{3}\left[32\sqrt{3} - 160\pi + 64\pi\sqrt{3}\right]$.

Proof. Let $T = \triangle ABC$ with $\angle A = 90°$, $\angle B = 60°$, and $\angle C = 30°$. Let a, b, and c be the radii of the circles centered at A, B, and C respectively. Now

$$\begin{cases} a + b &= 8, \\ a + c &= 8\sqrt{3}, \\ b + c &= 16. \end{cases}$$

By adding these three equations, we get $2a + 2b + 2c = 24 + 8\sqrt{3}$, so $a + b + c = 12 + 4\sqrt{3}$. Subtracting each equation in the original system from this sum, we have

$$(a, b, c) = (4\sqrt{3} - 4, 12 - 4\sqrt{3}, 4 + 4\sqrt{3}).$$

Now, the area of $\triangle ABC$ is $\frac{1}{2}(8)(8\sqrt{3}) = 32\sqrt{3}$. This triangle contains one quarter, one sixth, and one twelfth of the circles centered at A, B, and C respectively, so the desired area is

$$32\sqrt{3} - \frac{1}{4}(\pi a^2) - \frac{1}{6}(\pi b^2) - \frac{1}{12}(\pi c^2) = 32\sqrt{3} - \pi\left(\frac{64 - 32\sqrt{3}}{4} + \frac{192 - 96\sqrt{3}}{6} + \frac{64 + 32\sqrt{3}}{12}\right)$$

$$= \frac{1}{3}\left[32\sqrt{3} - 160\pi + 64\pi\sqrt{3}\right].$$

□

Source: Andreescu, Titu, and Jonathan Kane. *Purple Comet! Math Meet: The First Ten Years.* XYZ Press (2013), pp. 122-123.