University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #5

(Spring 2020)

Find all integers x and y for which $x^3 + x^2 + x + 1 = y^3$.

Solution:

The only integer solutions are $(x, y) = (0, 1)$ and $(x, y) = (-1, 0)$.

Proof. First of all,

$$\begin{aligned} 0 &< 3x^2 + (x-1)^2 + 1 \\ 0 &< 4x^2 - 2x + 2 \\ x^3 - 3x^2 + 3x - 1 &< x^3 + x^2 + x + 1 \\ (x-1)^3 &< x^3 + x^2 + x + 1 \end{aligned}$$

Secondly,

$$\begin{aligned} x \le -1 \quad &\text{or} \quad x \ge 0 \\ 2x(x+1) &\ge 0 \\ 2x^2 + 2x &\ge 0 \\ x^3 + 3x^2 + 3x + 1 &\ge x^3 + x^2 + x + 1 \\ (x+1)^3 &\ge x^3 + x^2 + x + 1 \end{aligned}$$

So if $x^3 + x^2 + x + 1 = y^3$, then $(x-1)^3 < y^3 \le (x+1)^3$, which means $y = x$ or $y = x + 1$.

However, if $y = x$, then $x^2 + x + 1 = 0$, and then x is not an integer, nor even a real number, since the discriminant of this quadratic equation is $b^2 - 4ac = -3 < 0$. It follows that $y = x + 1$, and (by the above computation) that

$$\begin{aligned} (x+1)^3 &= x^3 + x^2 + x + 1 \\ x(x+1) &= 0 \\ x = 0 \quad &\text{or} \quad x = -1. \end{aligned}$$

□

Source: Furdui, Ovidiu. "Quickies 980." *Mathematics Magazine* **81**:2 (April 2008), pp. 156, 161.