UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #3

(Spring 2020)

Of all the rational numbers in the open interval $\left(\frac{2018}{2019}, \frac{2019}{2020}\right)$, find the one with the smallest positive denominator.

Solution:

The desired rational number is $\boxed{\frac{4037}{4039}}$.

Proof. It is easy to check that

$$\frac{2018}{2019} = \frac{4036}{4038} = 1 - \frac{2}{4038} < 1 - \frac{2}{4039} < 1 - \frac{2}{4040} = \frac{4038}{4040} = \frac{2019}{2020}.$$

Thus $1 - \frac{2}{4039} = \frac{4037}{4039}$ is in the given interval.

Conversely, let p and q be integers for which $q > 0$ and $\frac{p}{q}$ is in the given interval. We have $\frac{p}{q} < \frac{2019}{2020} < 1$, so $p < q$. Moreover,

$$\begin{array}{rcccl} \frac{2018}{2019} & < & \frac{p}{q} & < & \frac{2019}{2020} \\ \frac{-1}{2019} & < & \frac{p}{q} - 1 & < & \frac{-1}{2020} \\ \frac{1}{2019} & > & \frac{q-p}{q} & > & \frac{1}{2020} \\ \frac{1}{2019(q-p)} & > & \frac{1}{q} & > & \frac{1}{2020(q-p)} \\ 2019(q-p) & < & q & < & 2020(q-p) \end{array}$$

□

This implies that $q - p \neq 1$ (since q cannot be an integer between 2019 and 2020). But we know that $q - p$ is a positive integer, so $q - p \geq 2$. Therefore,

$$4038 = 2019 \cdot 2 \leq 2019(q-p) < q.$$

It follows that $q \geq 4039$.

Source: Michael Black, suggested by Wissner-Gross, Zach. "Can You Find A Number Worth Its Weight In Letters?" *The Riddler* (`https://fivethirtyeight.com/features/can-you-find-a-number-worth-its-weight-in-letters/`), 10 January 2020.