University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #2

(Spring 2020)

Prove that for any pair of relatively prime whole numbers x and n, there is some whole number t for which $1 + x + \cdots + x^t$ is a multiple of n.
For example, if $x = 168$ and $n = 25$, we can find $1 + x + x^2 + x^3 = 4{,}770{,}025$.

Solution:

Proof. Let $a_k = \sum_{i=0}^{k} x^i = 1 + x + \cdots + x^k$. The set $A = \{a_0, \ldots, a_n\}$ contains $n + 1$ whole numbers, so by the pigeonhole principle, there must be two elements of A — call them a_b and a_c, with $b < c$ — that leave the same remainder when divided by n. Then $a_c - a_b$ is a multiple of n. But

$$a_c - a_b = x^{b+1} + \cdots + x^c = x^{b+1}(1 + \cdots + x^{c-b-1}) = x^{b+1} a_{c-b-1},$$

and since x is relatively prime to n, a_{c-b-1} must be a multiple of n. In other words, $1 + x + \cdots + x^t$ is a multiple of n when $t = c - b - 1$. □

Source: Michael Reid. Solution of “Multiples Without Large Digits,” posed by Gregory Galperin and Yury J. Ionin. *American Mathematical Monthly* **126**:10 (December 2019), 950.