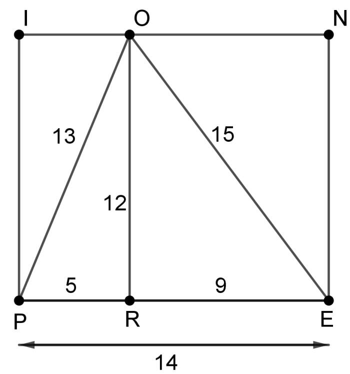
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Spring 2019)

In the figure below (not to scale), the sides of $\triangle POE$ and its height PI are four consecutive integers. Find the area of rectangle $PINE$.



Solution:
Let $h = PI$, and let $a < b < c$ be the sides of $\triangle POE$. We can see that $PO > h$ and $EO > h$, so $h < b < c$.

- Suppose for the sake of contradiction that $a < h$. By the Pythagorean theorem,

$$h - 1 = a = PE = PR + RE = \sqrt{2h+1} + \sqrt{4h+4}.$$

Simplifying, we get $h^3 - 16h^2 + 24h + 16 = 0$, but this equation has no integer solutions.

We conclude that $h < a$, so $(h, a, b, c) = (b-2, b-1, b, b+1)$. Let $p \in \{a, b, c\}$ be the length of the side perpendicular to the height h. By Heron's formula, we have:

$$\begin{aligned} \frac{1}{2}ph &= \sqrt{\left(\frac{3b}{2}\right)\left(\frac{b+2}{2}\right)\left(\frac{b}{2}\right)\left(\frac{b-2}{2}\right)} \\ 4p^2(b-2)^2 &= 3b^2(b+2)(b-2) \\ 4p^2(b-2) &= 3b^2(b+2) \end{aligned}$$

- Suppose for the sake of contradiction that $p = a = b - 1$. Simplifying, we get $b^3 - 22b^2 + 20b - 8 = 0$, but this has no integer solutions.
- Suppose for the sake of contradiction that $p = c = b + 1$. Simplifying, we get $b^3 - 6b^2 + 20b - 8 = 0$, but this has no integer solutions.

Thus $p = b$, and since $b \neq 0$, we have $4b - 8 = 3b + 6$, or $b = 14$. Thus $h = 12$, and the area of rectangle $PINE$ is $14 \cdot 12 = \boxed{168}$.

Source: Inspired by:

[Dud67] Henry Ernest Dudeney, *230: find the triangle*, 536 puzzles and curious problems, 1967, pp. 71,278.