University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Spring 2019)

For any integer $n \geq 1$, let $f(n)$ denote the number of times that the digit 2 appears in the integers from 1 through n. For example, $f(32) = 14$, because the digit 2 appears in the natural numbers $\{2, 12, 20, 21, 22 \text{ (twice)}, 23, 24, 25, 26, 27, 28, 29, 32\}$.

Find an integer n with the property that $f(n) = n$.

Bonus challenge: Are there infinitely many values of n for which $f(n) = n$?

Solution:

One such integer is $n = 10^{10}$.

Proof. Of the n integers between 1 and n, exactly $\frac{1}{10}$ of them have a 2 in any given decimal place. Since these integers contain 10 decimal places (including leading zeros where necessary, and ignoring the extra digit in n, which is not a 2), we get $f(n) = 10(n/10) = n$. □

Solution to bonus challenge. There are only finitely many values of n with $f(n) = n$. To see this, let $N = 10^{20}$. By the method we used above, we compute $f(N) = 20(N/10) = 2N$. Now suppose $x > N$. Divide x by N to obtain integers q and r with $q \geq 1$ and $0 \leq r < N$ such that $x = qN + r$. Because N is a power of 10, the last N integers from $((q-1)N + 1)$ to qN contain all the digits that appear in the N integers from 1 to N, plus extras. Thus $f(qN) \geq f((q-1)N) + f(N)$. By induction, we have $f(qN) \geq qf(N)$. Hence

$$f(x) \geq f(qN) \geq qf(N) = q(2N) = qN + qN \geq qN + N > qN + r = x.$$

So if $f(x) = x$, then $x \leq N$. Hence only finitely many solutions exist. □

Source: Treviño, Enrique. "Problem 2062." *Mathematics Magazine* **92**:1 (February 2019), p. 72.