University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #8

(Spring 2019)

When L is a list of distinct terms, a *permutation* of L is a list of the same length with the same terms.
For example, 314265 is a permutation of 123456 with the property that none of the first 4 terms is greater than 4.
Find the number of permutations of 123456 such that for each k with $1 \leq k < 6$, at least one of the first k terms of the permutation is greater than k.

Solution:

The number is 461.

Proof. We say a permutation of $1 \cdots n$ is *good* if for each k with $1 \leq k < n$, at least one of the first k terms of the permutation is greater than k; otherwise the permutation is *bad*. Let g_n be the number of good permutations of $1 \cdots n$, and let b_n be the number of bad permutations of $1 \cdots n$. Our problem is to find g_6.
We know the following:

- Vacuously, $g_1 = 1$ and $b_1 = 0$.
- For each n, we have $g_n + b_n = n!$.

Given a bad permutation P of $1 \cdots n$, let $m = m(P)$ be the *least* natural number for which the first m terms of P are less than or equal to m. For each k with $1 \leq k < n$, we can count the bad permutations of $1 \cdots n$ with $m = k$: since m is minimal, the first k terms of such a permutation form a good permutation of $1 \cdots k$, so they can be chosen in g_k ways, while the remaining $n - k$ terms can be chosen in $(n-k)!$ ways. Thus, for each n, we have $b_n = \sum_{k=1}^{n-1} g_k \cdot (n-k)!$, and therefore

$$g_n = n! - \sum_{k=1}^{n-1} g_k (n-k)!$$

n	g_n		
2	$2! - g_1(1!)$	$= 2 - (1)(1)$	$= 1$
3	$3! - g_1(2!) - g_2(1!)$	$= 6 - (1)(2) - (1)(1)$	$= 3$
4	$4! - g_1(3!) - g_2(2!) - g_3(1!)$	$= 24 - (1)(6) - (1)(2) - (3)(1)$	$= 13$
5	$5! - g_1(4!) - g_2(3!) - g_3(2!) - g_4(1!)$	$= 120 - (1)(24) - (1)(6) - (3)(2) - (13)(1)$	$= 71$
6	$6! - g_1(5!) - g_2(4!) - g_3(3!) - g_4(2!) - g_5(1!)$	$= 720 - (1)(120) - (1)(24) - (3)(6) - (13)(2) - (71)(1)$	$= 461.$

□

Source: Problem 11 of the 2018 American Invitational Mathematics Exam.