UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Spring 2019)

Evaluate:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

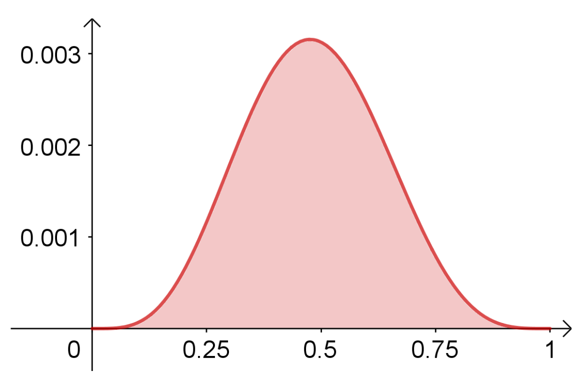
Solution:
The integral equals the error in a common approximation of π: $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$

Proof. Expand $x^4(1-x)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$, and apply long division:

$$\begin{array}{r}
x^6 - 4x^5 + 5x^4 - 4x^2 + 4 \\
x^2+1 \overline{\smash{\big)}\; x^8 - 4x^7 + 6x^6 - 4x^5 \quad + x^4 \qquad\qquad} \\
\underline{-x^8 \qquad\quad - x^6} \qquad\qquad\qquad\qquad \\
-4x^7 + 5x^6 - 4x^5 \qquad\qquad\qquad \\
\underline{4x^7 \qquad\quad + 4x^5} \qquad\qquad\qquad \\
5x^6 \qquad\quad + x^4 \qquad\quad \\
\underline{-5x^6 \qquad\quad - 5x^4} \qquad\quad \\
-4x^4 \qquad\quad \\
\underline{4x^4 + 4x^2} \qquad \\
4x^2 \qquad \\
\underline{-4x^2 - 4} \\
-4
\end{array}$$

Now integrate:

$$\begin{aligned}
&\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \\
=\;& \int_0^1 \left[x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right] dx \\
=\;& \left[\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} + 4x - 4\arctan x\right]_0^1 \\
=\;& \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi \\
=\;& \frac{22}{7} - \pi.
\end{aligned}$$



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Happy π Day!

Source: Problem A1 of the 1968 William Lowell Putnam Mathematics Competition, suggested by Kirthi Premadasa.