University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Spring 2019)

Nine is the smallest number that can be written as the sum of one or more consecutive positive integers in three different ways:

$$2 + 3 + 4 = 4 + 5 = 9$$

In how many ways can 2019 be written as the sum of one or more consecutive positive integers?

Solution:
There are exactly four ways:

$$334 + 335 + 336 + 337 + 338 + 339 = 672 + 673 + 674 = 1009 + 1010 = 2019$$

These sums are in one-to-one correspondence with the positive divisors of 2019, which are $\{1, 3, 673, 2019\}$.
More generally, let n be a positive integer, and write $n = 2^k m$ where m is odd. Define $C = \frac{-1+\sqrt{8n+1}}{2}$. Suppose d is a divisor of m.

- If $d \le C$, then d corresponds to the sum of (odd) length d whose terms have average value n/d. [Thus the middle term in the sum is the integer n/d.]
- If $d > C$, then d corresponds to the sum of (even) length $2n/d$ whose terms have average value $d/2$. [Thus the middle terms in the sum are the integers $\frac{d-1}{2}$ and $\frac{d+1}{2}$.]

This gives every sum of consecutive positive integers that equals n, because the correspondence is invertible:

- Given a sum of odd length d, we know that n is the product of d and the middle term in the sum, so $d|n$. But $\gcd(d,2) = 1$, so $d|m$. Also, the middle term is $\frac{n}{d}$ and is the $\left(\frac{d+1}{2}\right)^{\text{th}}$ term in the sum, so

$$\frac{n}{d} \ge \frac{d+1}{2} \quad \Rightarrow \quad n \ge \frac{d^2+d}{2} \quad \Rightarrow \quad 2n + \frac{1}{4} \ge \left(d + \frac{1}{2}\right)^2 \quad \Rightarrow \quad d \le C.$$

- Given a sum of even length $2k$, let $d = n/k$. Then the average value of the terms is $n/2k = d/2$, and since the average value is between the two middle terms, it is not an integer, so d is odd. Thus $d|m$. Also, since the k^{th} term is $\frac{d-1}{2}$, we have

$$k \le \frac{d-1}{2} \quad \Rightarrow \quad \frac{n}{d} \le \frac{d-1}{2} < \frac{d+1}{2} \quad \Rightarrow \quad 2n + \frac{1}{4} < \left(d + \frac{1}{2}\right)^2 \quad \Rightarrow \quad d > C.$$