University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #3

(Spring 2019)

Find all integers k for which $0 \le k \le 99$ and $\binom{99}{k}$ is not divisible by 3.

Solution:
The k-values are $\{0, 9, 18, 81, 90, 99\}$.

Proof. We will be working with polynomials with integer coefficients. Given two such polynomials f and g, we shall write $f \equiv g$ when every coefficient of $f - g$ is divisible by 3. Notice that for any f and g, we have $(f+g)^3 = f^3 + 3f^2g + 3fg^2 + g^3$, so $(f+g)^3 \equiv f^3 + g^3$. This fact is called a "Frobenius automorphism," or the "freshman's dream theorem." Applying it repeatedly, we have $f^9 + g^9 \equiv (f^3 + g^3)^3 \equiv ((f+g)^3)^3 = (f+g)^9$, and similarly with any power of 3 as the exponent. Therefore,

$$\begin{aligned} \sum_{k=0}^{99} \binom{99}{k} x^k &= (x+1)^{99} \\ &= (x+1)^{81}((x+1)^9)^2 \\ &\equiv (x^{81}+1)(x^9+1)^2 \\ &= x^{99} + 2x^{90} + x^{81} + x^{18} + 2x^9 + 1. \end{aligned}$$

In other words, the remainder when $\binom{99}{k}$ is divided by 3 is $\begin{cases} 1 & k = 0, 18, 81, 99; \\ 2 & k = 9, 90; \\ 0 & \text{otherwise.} \end{cases}$ □

Source: Sorel, Julien, in *Mathematics Magazine* **91**:1 (2-2018), 72, 77.