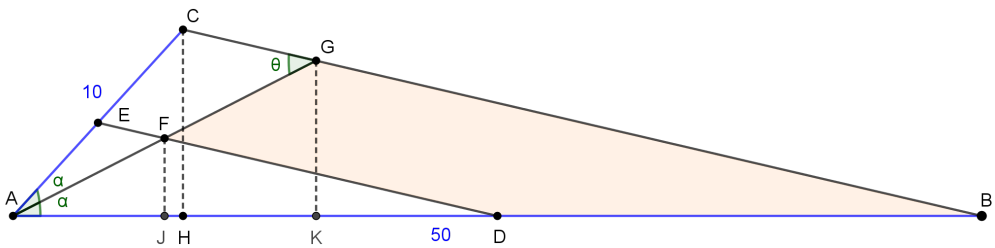
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #1

(Spring 2019)

Suppose $\triangle ABC$ has area 120, with $|AB| = 50$ and $|AC| = 10$. Let D be the midpoint of AB, and let E be the midpoint of AC. Suppose that the bisector of $\angle BAC$ intersects DE at F and intersects BC at G. Find the area of the quadrilateral $BDFG$.



Solution:
The area is 75.

Proof. Let θ denote the measure of $\angle AGC$, so that $180° - \theta$ is the measure of $\angle AGB$, and let α denote half the measure of $\angle BAC$. Use the law of sines twice to obtain:

$$\frac{10 \sin \alpha}{|CG|} = \sin \theta = \sin(180° - \theta) = \frac{50 \sin \alpha}{|BG|}.$$

It follows that $|BG| = 5|CG|$. (This is the "angle bisector theorem.")
Drop altitudes from C, F, and G to AB with feet at H, J, and K respectively. Starting with the given area of $\triangle ABC$, we use similarity of triangles:

$$\begin{aligned} 120 &= \frac{1}{2}(50)|CH| \\ 4.8 &= |CH| \\ |GK| &= \tfrac{5}{6}|CH| = 4 \\ |FG| &= \tfrac{1}{2}|GK| = 2 \end{aligned}$$

We can now find the area of $BDFG$ by subtracting the area of $\triangle ADF$ from the area of triangle ABG, obtaining

$$\frac{1}{2}(50)(4) - \frac{1}{2}(25)(2) = 100 - 25 = \boxed{75}.$$

□