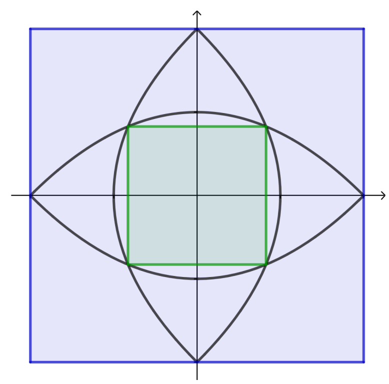
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #10

(Spring 2018)

Let S be a square, and let R be the set of points in S that are closer to the center of S than to any edge of S. Find $\dfrac{\text{area}(R)}{\text{area}(S)}$.

Solution:



We choose a coordinate system so that S has edges where $x = \pm 1$ and where $y = \pm 1$.
The parabola with vertex $(0, 0)$ and directrix $y = -1$ forms part of the boundary of R. This parabola passes through $(0, -1/2)$ and $(\pm 1, 0)$, so it is $y = (x^2 - 1)/2$. By symmetry, the boundary of R is formed by the parabolas $y = \pm(x^2 - 1)/2$ and $x = \pm(y^2 - 1)/2$.
By symmetry, these curves intersect in pairs at points where $y = \pm x$; the intersections are at $(\pm(\sqrt{2} - 1), \pm(\sqrt{2} - 1))$. The portion of R where $1 - \sqrt{2} \leq x \leq \sqrt{2} - 1$, and the portion where $1 - \sqrt{2} \leq y \leq \sqrt{2} - 1$, both have area

$$\int_{1-\sqrt{2}}^{\sqrt{2}-1} (1 - x^2)\, dx = \frac{8 - 4\sqrt{2}}{3}.$$

The union of those two regions is R, and their intersection is a square with side length $2\sqrt{2} - 2$, so the area of R is

$$2 \cdot \frac{8 - 4\sqrt{2}}{3} - (2\sqrt{2} - 2)^2 = \frac{16\sqrt{2} - 20}{3}.$$

Therefore,

$$\frac{\text{area}(R)}{\text{area}(S)} = \frac{4\sqrt{2} - 5}{3} \approx 0.21895.$$

Source:

[1] Nicholas R. Baeth, Loren Luther, and Rhonda McKee, *The downtown problem: variations on a Putnam problem*, Math. Mag. **90** (2017), no. 4, 243–257.

[2] Leonard F. Klosinski, Gerald L. Alexanderson, and Loren C. Larson, *The Fiftieth William Lowell Putnam Mathematical Competition*, Amer. Math. Monthly **98** (1991), no. 4, 319–327.