UNIVERSITY OF WISCONSIN PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #8

(Spring 2018)

Happy π Day! In the multiplication problem below, each character (P, I, $*$, ~, #) stands for a single digit (and no two characters represent the same digit). Find the digit that each character represents.

$$\begin{array}{cccc} & \mathtt{P} & * & \mathtt{I} \\ & \mathtt{P} & \mathtt{I} & \sim \\ \hline 3 & \mathtt{P} & \# & \mathtt{P} & * \end{array}$$

Solution:
The only solution is $(P, I, *, \sim, \#) = (1, 8, 6, 7, 4)$.

Proof. Write $a = \text{P}*\text{I}$ (that is, $a = 100P + 10 * + I$), $b = \text{PI} \sim$, and $c = ab = 3\text{P}\#\text{P}*$.
We see that $100P < a < 100(P+1)$ and $100P < b < 100(P+1)$, so $10000P^2 < c < 10000(P+1)^2$.
Thus $P^2 < \frac{c}{10000} < (P+1)^2$. Since $3 < \frac{c}{10000} < 4$, and $P \in \mathbb{N}$, we conclude that $P = 1$.
Now, since $a < 200$, we have $31000 < ab < 200b$, so $b > \frac{31000}{200} = 155$, so $I \geq 5$. Likewise, $a > 155$, so $* \geq 5$.
Because $31000 < ab < 32000$, we can compute that $(I, *)$ is one of the following pairs:

$$(5,9), (6,8), (6,9), \cancel{(7,7)}, (7,8), (8,6), (8,7), (9,5), (9,6)$$

Finally, $*$ is the ones digit of $(I)(\sim)$, so the possible triples $(I, \sim, *)$ are:

$$\cancel{(5,-,9)}, (6,3,8), \cancel{(6,8,8)}, \cancel{(6,-,9)}, (7,4,8), (8,2,6), (8,7,6), \cancel{(8,-,7)}, \cancel{(9,5,5)}, (9,4,6)$$

These triples represent the following products:

$(I, \sim, *)$	a	b	ab
$(6,3,8)$	186	163	30318
$(7,4,8)$	187	174	32538
$(8,2,6)$	168	182	30576
$(8,7,6)$	168	187	31416
$(9,4,6)$	169	194	32786

Only the fourth triple gives us a c value of the right size:

$$\begin{array}{ccccc} & 1 & 6 & 8 \\ & 1 & 8 & 7 \\ \hline 3 & 1 & 4 & 1 & 6 \end{array}$$

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