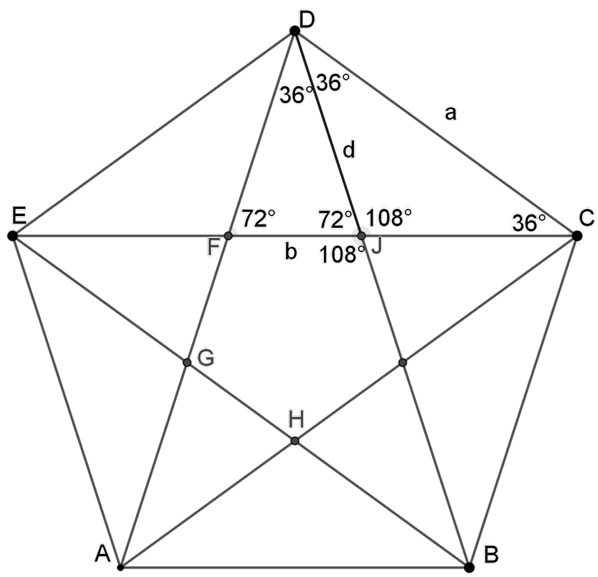
University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #7

(Spring 2018)

When you draw straight line segments joining all vertices of a regular pentagon P, you make a smaller regular pentagon Q inside. If P has sides of length a and Q has sides of length b, show that the distance d from a vertex of P to the nearest vertices of Q is $\sqrt{ab}$.



Solution:

Proof. The angles of a regular pentagon measure 108°. This means that the isosceles triangle $\triangle DFJ$ has base angles 72° and vertex angle 36°, while isosceles triangle $\triangle CDJ$ has vertex angle 108° and base angles 36°.

Applying the law of sines to these triangles, we get $\dfrac{a}{\sin 108°} = \dfrac{d}{\sin 36°}$ and $\dfrac{d}{\sin 72°} = \dfrac{b}{\sin 36°}$. Thus

$$a \cdot b = \frac{d \sin 108°}{\sin 36°} \cdot \frac{d \sin 36°}{\sin 72°} = d^2 \cdot \frac{\sin 108°}{\sin 72°}.$$

But since $108 + 72 = 180$, we have $\sin 108° = \sin 72°$, which means $ab = d^2$, as desired. □

Remark. This means that $\dfrac{a}{d} = \dfrac{d}{b}$; in fact, these fractions equal the *golden ratio* $\phi = \dfrac{1+\sqrt{5}}{2}$.