University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #5

(Spring 2018)

The faces of n unit cubes have been painted either blue or orange independently at random, with blue and orange equally likely colors for each face.
What is the probability that these cubes can be lined up to make an "orange train" — a $1 \times 1 \times n$ box in which only orange faces are visible?

Solution:
The probability is $\dfrac{10^n - 3^{n-1}(7n+3)}{2^{6n}}$.

Proof. Each cube has 6 faces, so there are 2^6 equally probable ways to paint a cube.
If any cube has two adjacent blue faces, we won't be able to make an orange train.
Painted cubes without adjacent blue faces come in two types:

- "Type-I cubes" have fewer than two blue faces: there are 7 of these among our 2^6 outcomes.
- "Type-II cubes" have exactly two blue faces, which are opposite one another: there are 3 of these.

Altogether, there are 2^{6n} equally probable ways to paint our n cubes, and 10^n of these outcomes are *plausible* — they contain only cubes without adjacent blue faces.
If we use a type-II cube at the end of our $1 \times 1 \times n$ box, a blue face will be visible, so we can't make an orange train if we have more than $n-2$ type-II cubes. But that's the only thing that can go wrong: as long as we have a plausible outcome with no more than $n-2$ type-II cubes, we can hide all the blue faces and make an orange train.
Of the 10^n plausible outcomes, there are exactly 3^n that have n type-II faces and exactly $\binom{n}{n-1}3^{n-1}7 = 7n3^{n-1}$ that have $n-1$ type-II faces. The remaining $10^n - 3^{n-1}(7n+3)$ outcomes are the ones in which we can make an orange train. $\square$

Question. What is the answer if faces are painted orange with probability p?

Source: Benjamin V. C. Collins and James Swenson, suggested by problem #9 of the 2005 American Invitational Mathematics Exam.

[1] Scott A. Annin, *A gentle introduction to the American Invitational Mathematics Exam*, MAA Problem Books Series, Mathematical Association of America, Washington, DC, 2015.