University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #4

(Spring 2018)

There are 3 ways to write 6 as a sum of two positive odd numbers:

1. 5+1
2. 3+3
3. 1+5

How many ways are there to write 2018 as a sum of 10 positive odd numbers?

Solution:
There are $\binom{1013}{9}$ ways, which is about 3 billion trillion.

Proof. We answer the more general question: In how many ways can we write the even integer $2n$ as a sum of m positive odd integers?
If m is odd, then any sum of m odd integers is odd, so there is no sum of m positive odd integers that equals $2n$, so from now on, we assume that m is even, and write $m = 2k$.
We seek non-negative integers x_i such that

$$\begin{aligned} 2n &= (2x_1+1)+(2x_2+1)+\cdots+(2x_{2k}+1) \\ 2n &= 2(x_1+x_2+\cdots+x_{2k})+2k \\ n &= x_1+x_2+\cdots+x_{2k}+k \\ n-k &= x_1+x_2+\cdots+x_{2k} \end{aligned}$$

So we need to know the number of ways to write $n-k$ as a sum of $2k$ non-negative integers. This can be represented as the number of arrangements containing $n-k$ stars and $2k-1$ bars: the bars divide the stars into groups. For example, if $n=3$ and $k=1$, we have 2 stars and 1 bar. Such arrangements correspond to ways of writing 2 as a sum of 2 non-negative integers, or writing 6 as a sum of 2 positive odd integers:

Arrangement	x_1+x_2	$(2x_1+1)+(2x_2+1)$
$**\|$	$2+0$	$5+1$
$*\|*$	$1+1$	$3+3$
$\|**$	$0+2$	$1+5$

In each arrangement, there are $n-k+2k-1 = n+k-1$ characters, so there are $\binom{n+k-1}{2k-1}$ arrangements. Taking $n=1009$ and $k=5$, we have

$$\binom{1009+5-1}{2\cdot5-1} = \binom{1013}{9} = \frac{1013\cdot1012\cdot1011\cdot1010\cdot1009\cdot1008\cdot1007\cdot1006\cdot1005}{9!} = \frac{1013\cdot(4\cdot253)\cdot(3\cdot337)\cdot(2\cdot505)\cdot1009\cdot(9\cdot8\cdot7\cdot2)\cdot1007\cdot1006\cdot(5\cdot3\cdot67)}{9!}$$

$$= 1013\cdot253\cdot337\cdot505\cdot1009\cdot1007\cdot1006\cdot67 = 2{,}987{,}064{,}948{,}892{,}522{,}439{,}590 \text{ arrangements.}$$

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