University of Wisconsin
Platteville
Department of Mathematics

Problem of the Week #3

(Spring 2018)

We say a function f has *width* d if there is a horizontal line segment of length d whose endpoints are both on the graph $y = f(x)$. For example, if $f(x) = x^3 - x$, then f has width 2 because $(-1, 0)$ and $(1, 0)$ are both on the graph of f.
Suppose that g is a continuous function with domain $(-\infty, \infty)$ that has both an absolute maximum and an absolute minimum. Is it true that g must have width d for every $d > 0$?

Solution:
It is true.

Proof. Let g be continuous on $(-\infty, \infty)$. Suppose that g has an absolute maximum at $x = a$ and an absolute minimum at $x = b$. Let $d > 0$.
Define $h(x) = g(x) - g(x - d)$. Then h is continuous on $(-\infty, \infty)$, $h(a) = g(a) - g(a - d) \geq 0$, and $h(b) = g(b) - g(b - d) \leq 0$. Therefore, by the intermediate value theorem, there is some c between a and b such that $h(c) = 0$. Then $g(c) = g(c - d)$, so the line segment from $(c - d, g(c - d))$ to $(c, g(c))$ is horizontal and d units long. This shows that g has width d. □

Source: Mortini, Raymond. "Quickies 1075." *Mathematics Magazine* **90**:5 (December 2017), pp. 384, 393.