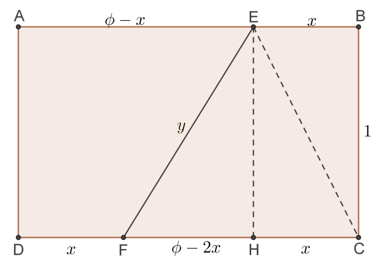
UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

PROBLEM OF THE WEEK #2

(Spring 2018)

The longer edge of a *golden rectangle* is ϕ times as long as the shorter edge, where $\phi = \frac{1+\sqrt{5}}{2}$. If you fold a golden rectangle so that its opposite corners touch, what is the ratio of the length of the crease to the length of the longer edge?

Solution:



First notice that $\phi^2 = \phi + 1$: we'll use this fact later.
Let $ABCD$ be a golden rectangle, with $AB = \phi$ and $BC = 1$. When we fold this rectangle so that A lies on C, the fold line is the perpendicular bisector of AC, so it passes through the center of the rectangle, intersecting AB at E and CD at F. Let $x = |BE|$. Since E is on the fold, we have $CE = AE = \phi - x$. By the Pythagorean theorem,

$$x^2 + 1 = (\phi - x)^2 = \phi^2 - 2\phi x + x^2 = \phi + 1 - 2\phi x + x^2.$$

It follows that $\phi = 2\phi x$, so $x = \frac{1}{2}$. By symmetry, we have $|DF| = \frac{1}{2}$. The perpendicular to CD through E meets CD at H, and $|FH| = \phi - 1$.
Now let $y = |EF|$ be the length of the fold. By the Pythagorean theorem,

$$y^2 = 1 + (\phi - 1)^2 = 1 + \phi^2 - 2\phi + 1 = 2 + (\phi + 1) - 2\phi = 3 - \phi.$$

It follows that $y = \sqrt{3 - \phi}$, and

$$\frac{y}{\phi} = \sqrt{\frac{3 - \phi}{\phi^2}} = \sqrt{\frac{5 - \sqrt{5}}{2} \cdot \frac{2}{3 + \sqrt{5}}} = \sqrt{\frac{(5 - 5\sqrt{5})(3 - \sqrt{5})}{4}} = \sqrt{5 - 2\sqrt{5}} \approx 0.72654.$$

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