University of Wisconsin Platteville
Department of Mathematics

Problem of the Week #1

(Spring 2018)

The perfect powers form a multiset

$$P = \{2^2, 2^3, 2^4, \ldots, 3^2, 3^3, 3^4, \ldots, 4^2, 4^3, 4^4, \ldots\}.$$

A multiset is just like a set, except that elements can occur multiple times. For example, $64 = 2^6 = 4^3 = 8^2$, so 64 appears three times in P.

Find the sum of the reciprocals of the perfect powers: $\displaystyle\sum_{n \in P} \frac{1}{n}$.

To be clear, the term $\frac{1}{64}$ appears three times in this sum.

Solution:

Using the geometric series formula and partial fractions, we can turn the problem into a telescoping sum:

Proof.

$$\begin{aligned}
\sum_{n \in P} \frac{1}{n} &= \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k^m} \\
&= \sum_{k=2}^{\infty} \frac{1/k^2}{1 - (1/k)} \\
&= \sum_{k=2}^{\infty} \frac{1}{k^2 - k} \\
&= \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \lim_{N \to \infty} \sum_{k=2}^{N} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \lim_{N \to \infty} 1 - \frac{1}{N} \\
&= \boxed{1}.
\end{aligned}$$

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