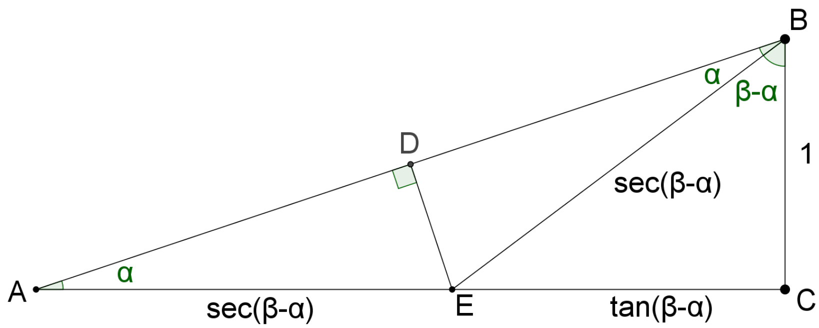
University of Wisconsin
PLATTEVILLE
Department of Mathematics

Problem of the Week #10

(Spring 2017)

Prove: If α and β are complementary angles, then $\tan(\beta - \alpha) + \sec(\beta - \alpha) = \tan\beta$.

Solution:



Proof. The figure provides a proof in the case that α and β are both acute, with $\beta > \alpha$. Let $\triangle ABC$ be a right triangle with angles α at A and β at B, scaled so that $|BC| = 1$. The perpendicular bisector of AB crosses AC at E (since $\alpha < \beta$). By the SAS theorem, $\angle ABE = \alpha$, so $\angle EBC = \beta - \alpha$. Thus $|EC| = \tan(\beta - \alpha)$ and $|BE| = \sec(\beta - \alpha)$. Finally, note that $\angle ABE = \angle BAE$, so $\triangle ABE$ is isosceles, and $|AE| = |BE| = \sec(\beta - \alpha)$. Hence

$$\sec(\beta - \alpha) + \tan(\beta - \alpha) = |AE| + |EC| = |AC| = \tan\beta.$$

□

Alternate proof.

$$\begin{aligned}
\tan(\beta - \alpha) + \sec(\beta - \alpha) &= \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} + \frac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\csc\alpha\csc\beta + \sec\alpha\sec\beta} \\
&= \frac{\tan\beta - \cot\beta}{1+1} + \frac{\sec^2\beta\csc^2\beta}{2\sec\beta\csc\beta} \\
&= \frac{1}{2}\left[\frac{\sin\beta}{\cos\beta} - \frac{\cos\beta}{\sin\beta} + \frac{1}{\sin\beta\cos\beta}\right] \\
&= \frac{\sin^2\beta - \cos^2\beta + 1}{2\sin\beta\cos\beta} \\
&= \frac{2\sin^2\beta}{2\sin\beta\cos\beta} \\
&= \tan\beta.
\end{aligned}$$

□

Source: Inspired by Francisco Javier García Capitán, "Proof Without Words: Tangents of 15 and 75 Degrees." *College Mathematics Journal* **48**:1 (January 2017), 35.