UNIVERSITY OF WISCONSIN PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #9

(Spring 2017)

You've just become head coach of the Pioneer football team. Things aren't looking good: you're trailing UW-Whitewater 14 – 0 in the fourth quarter. Suddenly, the Warhawks fumble — it's recovered by the Pioneers, and returned for a touchdown, with 1:24 left in the game! The score is 14 – 6. Should you kick the extra point, or go for two?

Details: Suppose that your kicker always makes extra points; let p be your team's success rate on two-point conversions. Which choice gives the Pioneers a better chance of winning the game? Assume that there's not enough time left for more than one more score before the game ends, and that the teams are equally likely to win if the game goes into overtime. Your choice of strategy should depend on the value of p.

Solution:

You should go for two if you think the probability that you'll be stopped is less than the *golden ratio conjugate* $\dfrac{-1+\sqrt{5}}{2} \approx 61.8\%$.

Proof. We might as well assume the Pioneers will score a second touchdown, and Whitewater won't score again: otherwise, the game is lost no matter what strategy is used.

Let's first examine what happens if the Pioneers are behind by 8 points when they score their final touchdown. In that case, they can't win unless they go for two. When they go for two, they tie the game with probability p, and ultimately win with probability $p/2$.

If the Pioneers are behind by 7 when they score their final touchdown, they can either kick, and win with probability $1/2$, or go for two, and win with probability p. Thus their win probability is $\max\{1/2, p\}$.

If the Pioneers are behind by 6 when they score their final touchdown, they can either kick, and win with probability 1, or go for two, and win with probability p. Thus their win probability is 1.

Suppose now that the Pioneers try the kick after their first touchdown. They end up behind by 7, which means their win probability is $W_1 = \max\{1/2, p\}$.

On the other hand, if the Pioneers go for two after their first touchdown, their win probability is $W_2 = p \cdot 1 + (1-p) \cdot p/2 = (3p - p^2)/2$.

Let $f(p) = W_2 - W_1$. If $1/2 \le p \le 1$, then $f(p) = p(1-p)/2$, which is positive, so the Pioneers should go for two after their first touchdown. On the other hand, if $0 \le p < 1/2$, then $f(p) = (-1 + 3p - p^2)/2$, which is positive iff $p > (3-\sqrt{5})/2$.

All in all, the Pioneers should go for two after the first touchdown iff $p > (3-\sqrt{5})/2 \approx 38.2\%$ — in other words, if their probability of failure $(1-p)$ is less than the golden ratio conjugate. □

Source: Urschel, John. "The Wednesday Morning Math Challenge: Week 11." November 23, 2016. (Note: John Urschel is an offensive lineman for the Baltimore Ravens and a Ph.D. student in mathematics at MIT.)
`http://www.theplayerstribune.com/the-wednesday-morning-math-challenge-week-11/`