University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

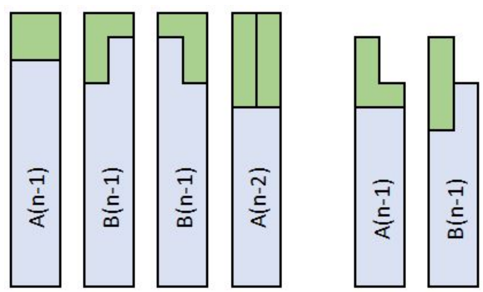
Problem of the Week #6

(Spring 2017)

In how many ways can a rectangle 2 units wide and 8 units high be tiled with Tetris blocks? Extra credit: Let n be an even integer. In how many ways can a rectangle 2 units wide and n units high be tiled with Tetris blocks?

Solution:
There are 25 ways to tile a 2×8 rectangle, and there are ${F_n}^2$ ways to tile a $2 \times 2n$ rectangle, where F_n is the n^{th} Fibonacci number.

Proof. Let $A(n)$ denote the number of ways to arrange n Tetris blocks into a rectangle two units wide.
Let $B(n)$ denote the number of ways to arrange n Tetris blocks into a region two units wide, so that one (specific) column is two units taller than the other.



By counting the ways to fill the top row of a region, we obtain the following recursions:

$$\begin{cases} A(n) &= A(n-1) + 2B(n-1) + A(n-2) \\ B(n) &= A(n-1) + B(n-1) \end{cases}$$

Let F_n denote the n^{th} Fibonacci number — thus $F_{-1} = 0$, $F_0 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for any integer $n > 0$. We claim that $A(n) = {F_n}^2$ and $B(n) = F_n F_{n-1}$.
The proof of the claim is by induction on n. The case $n = 0$ is clear: $A(0) = 1 = {F_0}^2$ and $B(0) = 0 = F_0 F_{-1}$.
Let $k \in \mathbb{N}$, and suppose the claim holds for all $n < k$. Then

$$A(n) = A(n-1) + 2B(n-1) + A(n-2) = {F_{n-1}}^2 + 2F_{n-1}F_{n-2} + {F_{n-2}}^2 = (F_{n-1} + F_{n-2})^2 = {F_n}^2$$

and

$$B(n) = A(n-1) + B(n-1) = {F_{n-1}}^2 + F_{n-1}F_{n-2} = F_{n-1}(F_{n-1} + F_{n-2}) = F_{n-1}F_n.$$

Specifically, the number of ways to tile a 2×8 rectangle is $A(4) = {F_4}^2 = 5^2 = 25$. □

Source: Wright, Bart. "Can You Win at Tetris?" In *Riddler Express* for 2017-01-27, `https://fivethirtyeight.com/features/can-you-win-at-tetris-can-you-save-a-species/`.
Solution suggested by Samuel Wirajaya at `http://tinyurl.com/TetrisTiling`.