University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #5

(Spring 2017)

Does the equation

$$a^2 + b^7 + c^{13} + d^{14} = e^{15}$$

have a solution in positive integers a, b, c, d, e?

Solution:
Yes: one such solution is $(A, B, C, D, E) = (4^{637}, 4^{182}, 4^{98}, 4^{91}, 4^{85})$.

Proof.

$$A^2 + B^7 + C^{13} + D^{14} = (4^{637})^2 + (4^{182})^7 + (4^{98})^{13} + (4^{91})^{14} = 4 \cdot 4^{1274} = 4^{1275} = (4^{85})^{15} = E^{15}.$$

□

Explanation. We'd like to take the logarithm of both sides of $A^2 + B^7 + C^{13} + D^{14} = E^{15}$, but can't simplify the left-hand side. But we're just looking for one solution: let's suppose that all four terms on the left-hand side are equal to N. Then we have $4N = E^{15}$. This suggests a base-4 logarithm: $1 + \log_4 N = 15 \log_4 E$. It would be ideal if N and E were powers of 4; say $N = 4^K$ and $E = 4^M$. Then we need $1 + K = 15M$. Ideally, K should be a multiple of 2, 7, 13, and 14.
We have $\operatorname{lcm}(2, 7, 13, 14) = 13 \cdot 14$, and $\gcd(13 \cdot 14, 15) = 1$. Thus there exist integers x and y such that $1 = (13 \cdot 14)x + 15y$. We can find such a pair by the reverse Euclidean algorithm: one example is

$$\begin{aligned}
(x, y) &= (-7, 85) \\
1 &= (13 \cdot 14)(-7) + (15)(85) \\
13 \cdot 14 \cdot 7 + 1 &= 15 \cdot 85 \\
4 \cdot 4^{13 \cdot 14 \cdot 7} &= 4^{15 \cdot 85} \\
4^{13 \cdot 14 \cdot 7} + 4^{13 \cdot 14 \cdot 7} + 4^{13 \cdot 14 \cdot 7} + 4^{13 \cdot 14 \cdot 7} &= 4^{15 \cdot 85} \\
(4^{13 \cdot 7 \cdot 7})^2 + (4^{13 \cdot 14})^7 + (4^{14 \cdot 7})^{13} + (4^{13 \cdot 7})^{14} &= (4^{85})^{15}.
\end{aligned}$$

□

Source:
Chu, Adrian. "Quickies 1065." *Mathematics Magazine* **89**:5 (December 2016), 379, 385.